Quadrant II – Transcript and Related Materials

Programme : Bachelor of Science (Third Year)

Subject : Electronics

Paper Code : ELD 105

Paper Title : Photonics

Unit : III

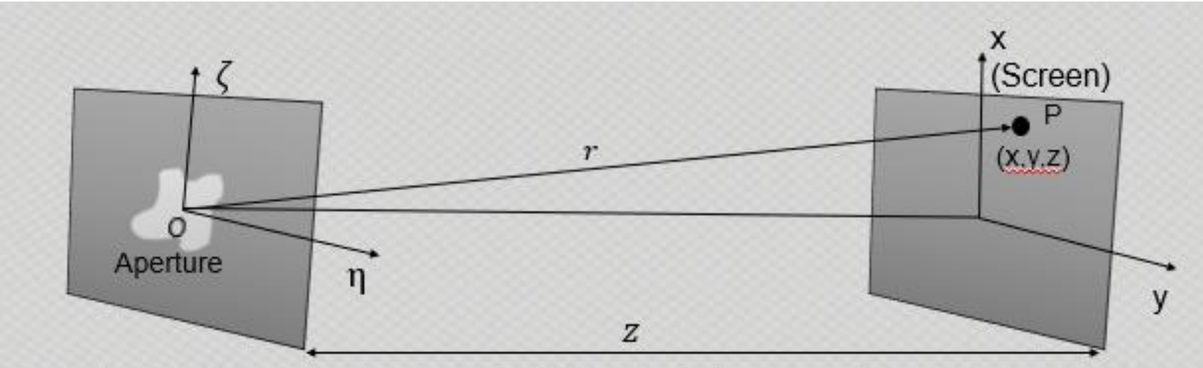
Module Name : Diffraction Integral, Fresnel and Fraunhoffer approximations

Module No : 14

Name of the Presenter : Ms. Froila Valency Rodrigues

Notes:

Diffraction Integral:



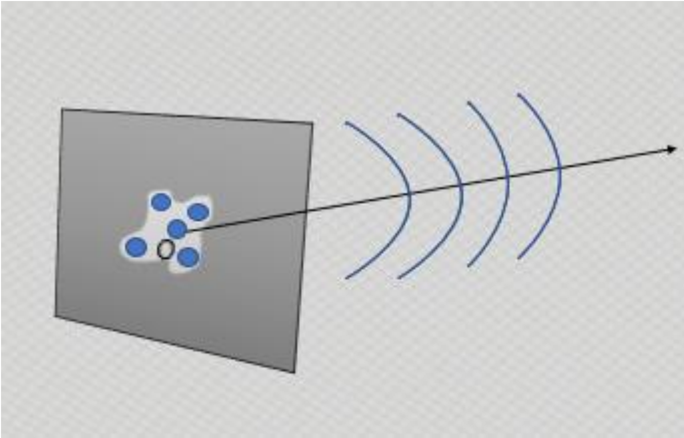
$r = distance\ between\ point\ source\ O\ and\ P(point\ of\ observation)$

$U(\zeta,\eta) = transmittance\ function\ limited\ to\ the\ aperture$

The transmittance function of the aperture is given as :

$U(\zeta,\eta) = 1 \quad Inside\ the\ aperture$

$U(\zeta,\eta) = 0 \quad Outside\ the\ aperture$



- Assuming the aperture is made up of many points, then according to Huygen's principle all the points act as secondary sources generating secondary spherical wavelets.
- The resultant of all spherical waves will be seen at point 'P' on the screen.
- $The\ diffraction\ at\ P\ is\ given\ by$
- $U(x,y) \propto \iint_{-\infty}^{\infty} U(\zeta,\eta)\ \left(\frac{e^{jkr}}{r}\right) d\zeta d\eta$
- $U(x,y) = \mathrm{C} \iint_{-\infty}^{\infty} U(\zeta,\eta)\ \left(\frac{e^{jkr}}{r}\right) d\zeta d\eta$ ________(*)
- $Where\ C\ is\ the\ constant\ we\ need\ to\ determine.$
- $k\ \rightarrow wave\ number,\ given\ as\ k = \frac{2\pi}{\lambda}$
- Considering no aperture, the plane wave will travel with no change.

$$U(\zeta,\eta) = 1$$

$$U(x,y) = \mathrm{C} \iint_{-\infty}^{\infty} 1.\ \left(\frac{e^{jkr}}{r}\right) d\zeta d\eta = e^{jkr} \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_(1)}$$

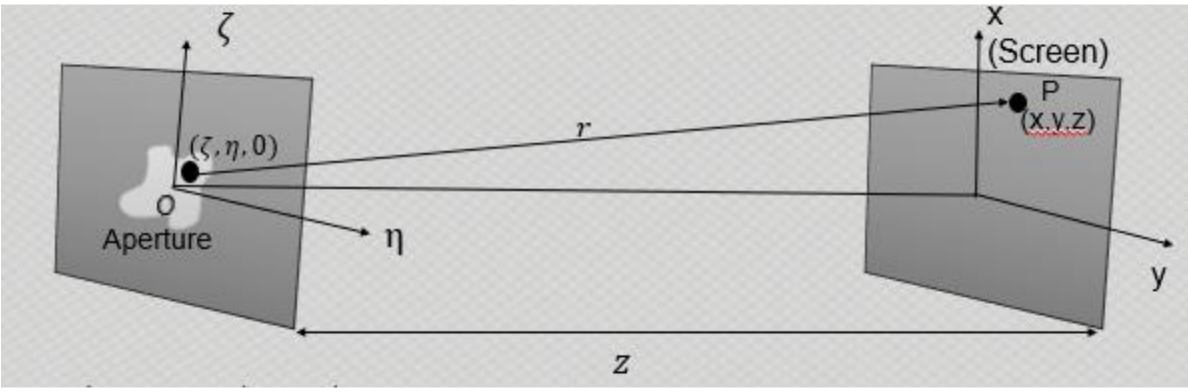
$$Where\ e^{jkr}\ is\ a\ plane\ wave$$

$$The\ distance\ between\ point\ O\ and\ P\ is\ given\ by:$$

$$r = \sqrt{(x-\zeta)^2 + (y-\eta)^2 + z^2}$$

$$Taking\ z\ common\ and\ outside\ square\ root$$

$$r = z\sqrt{1 + \left(\frac{x-\zeta}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2} \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(2)}$$



$$As\ the\ aperture\ is\ very\ small,\ we\ can\ say\ that\ \measuredangle\ , made\ by\ r\ with\ P\ is\ very\ small,$$

$$\therefore r \approx z$$

$$Expression\ for\ spherical\ waves\ is\ given\ as$$

$$\frac{e^{jkr}}{r} \cong \frac{e^{jkz}}{z}\exp\left\{\frac{jk}{2}\left[\left(\frac{x-\zeta}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2\right]\right\}$$

$$\frac{e^{jkr}}{r} \cong \frac{e^{jkz}}{z}\exp\left\{\frac{jk}{2z}[(x-\zeta)^2 + (y-\eta)^2]\right\} \_\_\_\_\_\_\_\_(6)$$

$Substituting\ for\ \frac{e^{jkr}}{r}$ in (1) $we\ get$

$$\mathrm{C}\frac{e^{jkz}}{z}\iint_{-\infty}^{\infty} e^{\frac{jk}{2z}(x-\zeta)^2} e^{\frac{jk}{2z}(y-\eta)^2} d\zeta d\eta = e^{jkz} \_\_\_\_\_\_\_\_(7)$$

$$Consider\ I_1$$

$$I_1 = \int_{-\infty}^{\infty} e^{\frac{jk}{2z}(x-\zeta)^2} d\,\zeta$$

$$Substitute\ (x-\zeta) = \gamma$$

$$differentiating\ w.r.t\ \zeta$$

$$d\zeta = -d\gamma$$

$$I_1 = -\int_{-\infty}^{\infty} e^{\frac{jk}{2z}\gamma^2} d\gamma$$

$$So\ comparing\ \alpha = -\frac{jk}{2z}\ , \beta = 0$$

$$= (-1)\sqrt{\frac{\pi}{-jk/2z}}\,e^{0}$$

$$= -\sqrt{\frac{2\pi z}{k}}\sqrt{\frac{1}{-j}}$$

You should know that

$$\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x}$$

$$= \sqrt{\frac{\pi}{\alpha}}\,e^{\frac{\beta^2}{4\alpha}}$$

$$Consider\ I_2$$

$$I_2 = \int_{-\infty}^{\infty} e^{\frac{jk}{2z}(y-\eta)^2}\,d\eta$$

$$Solution\ of\ this\ integral\ is$$

$$= -\sqrt{\frac{2\pi z}{k}}\sqrt{\frac{1}{-j}}$$

$$Substituting\ I_1 and\ I_2\ in\ (7)\ we\ get$$

$$C\,\frac{e^{jkz}}{z}\left(-\sqrt{\frac{2\pi z}{k}}\sqrt{\frac{1}{-j}}\right)^2 = e^{jkz}$$

$$C\left(\frac{2\pi}{-kj}\right) = 1,\ \ Where\ \lambda = \frac{2\pi}{k}$$

$$C\left(\frac{\lambda}{-j}\right) = 1 \Rightarrow C = -\frac{j}{\lambda} = \frac{1}{j\lambda}\_\_\_\_\_\_\_\_\_\_\_(8)$$

$So\ the\ final\ form\ of\ diffraction\ integral\ is\ given\ by\ substituting\ (8) in\ (*)$:

$$U(x,y) = \frac{1}{j\lambda}\iint_{-\infty}^{\infty} U(\zeta,\eta)\,\left(\frac{e^{jkr}}{r}\right)d\zeta d\eta\,\_\_\_\_\_\_\_\_\_\_\_(9)$$

=> Diffraction Integral

We had written

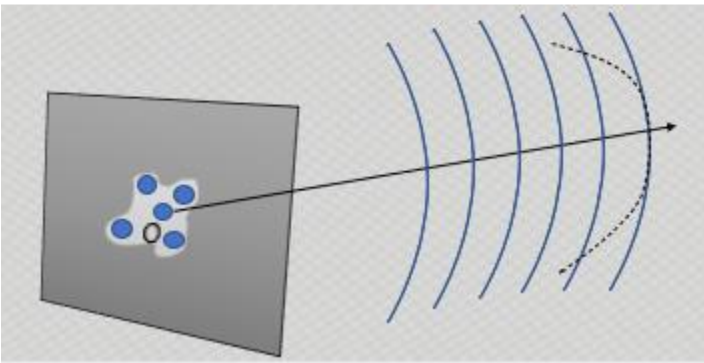
$$r = z + \frac{(x-\zeta)^2}{2z} + \frac{(y-\eta)^2}{2z} \quad \_\_\_\_\_\_\_\_\_(10)$$

Before Binomial approximation we had $r^2 = x^2 + y^2 + z^2$ *which is eq. of sphere in* $3D$.

Dropping higher order terms in Binomial approximation we get (10)

This is equation of parabaloid in $3D$ *or parabola in* $2D$.

Hence this approximation replaces a spherical wavefront by a paraboloid.



Substituting for r in eq. (9) *we get*

$$U(\mathrm{x},\mathrm{y}) = \frac{1}{\mathrm{j}\lambda}\iint_{-\infty}^{\infty} U(\zeta,\eta)\left(\frac{e^{jkz}}{z}\right) e^{\frac{jk}{2z}[(x-\zeta)^2+(y-\eta)^2]} d\zeta d\eta$$

$$U(\mathrm{x},\mathrm{y}) = \frac{e^{jkz}}{z\mathrm{j}\lambda}\iint_{-\infty}^{\infty} U(\zeta,\eta)\, e^{\frac{jk}{2z}[(x-\zeta)^2+(y-\eta)^2]} d\zeta d\eta \quad \_\_\_\_\_\_(9')$$

eq. $(9')$ *is in the form of convolution integral*

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$$Convolution\ form\ of\ final\ diffraction\ pattern\ on\ screen\ is:$$

$$h(x,y) * U\ (\zeta,\eta) \_\_\_\_\_\_\_\_\_\_\_\_ (11)$$

$Where\ U \rightarrow transmittance\ function\ of\ aperture$ given as eq. (9)

$$h(x,y) \rightarrow Fresnel's\ impulse\ response\ function\ given\ as$$

$$Exploring\ the\ property\ of\ Fourier\ transform$$

$$\mathcal{F}\{g * f\} = \mathcal{F}(g).\mathcal{F}(f) \_\_\_\_\_\_\_\_\_\_\_\_ (12)$$

$$Final\ diffraction\ pattern\ on\ the\ screen: \mathcal{F}(h).\mathcal{F}(U)$$

$$In\ the\ diffraction\ integral\ we\ have\ made\ approximations\ like$$

$|x-\zeta| \ll z,\ |y-\eta| \ll z$ and considered 1st two terms of Binomial approximation

$$Calculating\ the\ third\ term\ as \frac{n(n-1)x^2}{2!}\ we\ get\ the\ phase\ we\ dropped\ while$$

$$replacing\ spherical\ wavefronts\ with\ parabolic\ wavefronts.$$

Fresnel Approximation

$$we\ get\ the\ phase\ change\ as$$

$$kz^{-3}\left(\frac{1}{8}\right)[(x-\zeta)^2 + (Y-\eta)^2]^2 < 1\ rad \_\_\_\_\_\_\_\_\_\_\_\_ (13)$$

$$eq.\ (13)\ is\ the\ maximum\ phase\ change\ induced\ by\ dropping\ third\ term\ is\ less\ than\ 1\ rad$$

$$For\ fresnel\ condition\ to\ hold\ good\ eq.\ (13)\ should\ satisfy$$

$$i.e\ z \gg \sqrt[3]{\frac{\pi}{4\lambda}[(x-\zeta)^2 + (y-\eta)^2]^2 max} \_\_\_\_\_\_\_\_ (a)$$

$$So\ the\ diffraction\ integral\ is\ given\ by:$$

$$U(x,y) = \frac{e^{jkz}}{zj\lambda} e^{\frac{jk}{2z}(x^2+y^2)} \iint_{-\infty}^{\infty} U(\zeta,\eta)\ e^{\frac{jk}{2z}(\zeta^2+\eta^2)}.e^{-\frac{jk}{z}(x\zeta+y\eta)} d\zeta d\eta \_\_\_\_ (14)$$

$$Eq\ (14) is\ the\ product\ of\ the\ Fourier\ transform\ of\ Fresnel\ Impulse$$

$$Response\ x\ U(\zeta,\eta) \Rightarrow Fresnel\ diffraction\ integral\ at\ near\ field\ z$$

$$Eq.\ (a) is\ the\ Fresnel\ Approximation$$

Fraunhoffer Approximation

$$If\ z \gg \frac{k}{2}(\zeta^2+\eta^2)_{max} \Rightarrow Fraunhoffer\ Approximation\ for\ far\ field$$

$$e^{\frac{jk}{2z}(\zeta^2+\eta^2)} \approx 1 \_\_\_\_\_\_\_\_\_\_(15)$$

$$Putting\ eq.\ (15)\ in\ eq.\ (14) we\ get$$

$$U(x,y) = \frac{e^{jkz}}{zj\lambda} e^{\frac{jk}{2z}(x^2+y^2)} \iint_{-\infty}^{\infty} U(\zeta,\eta)\, e^{-\frac{jk}{z}(x\zeta+y\eta)} d\zeta d\eta \_\_\_\_\_(15)$$

$$Eq\ (15)\ is\ the\ \mathcal{F}\{U(\zeta,\eta)\} \Rightarrow Fraunhoffer\ diffraction\ integral$$

Diagrammatic view of the diffraction regimes: