Quadrant II – Transcript and Related Materials

Programme: Bachelor of Science

Subject: Mathematics

Paper Code: MTE103

Paper Title: Number Theory

Unit: 1

Module Name: LCM and its Properties

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Notes

Definition:

Least Common Multiple:

Let $a,\ b \in \mathbb{Z} \setminus \{0\}$. The least common multiple of a and b is the positive integer m satisfying

i. $a|m \textit{ and } b|m$

ii. If $a|c$ and $b|c$, with $c > 0$ then $m \leq c$.

Notation:

Least common multiple of a and b is denoted by $lcm(a, b)$.

Examples:

$lcm(12,4) = 12, lcm(15, -5) = 15, lcm(-6, -8) = 24$

- For positive integers a and b $\gcd(a, b)\ lcm(a, b) = ab$.

Proof: Let $d = \gcd(a, b)$, then $a = dr$ and $b = ds$ for integers r and s.

Let $m = \frac{ab}{d}$, then $m = as = br$.

Let c be a positive integer that is a common multiple of a and b.

i.e., let $c = au = bv$ for some integers u and v.

There exist integers x and y such that $d = ax + by$. Then

$$\frac{c}{m} = \frac{cd}{ab} = \frac{c(ax+by)}{ab} = \left(\frac{c}{b}\right)x + \left(\frac{c}{a}\right)y = vx + uy \in \mathbb{Z}$$

Therefore, $m|c$.

$\Rightarrow m \leq c$.

$\Rightarrow m = lcm(a,b)$

Hence, $lcm(a,b) = \frac{ab}{d} = \frac{ab}{\gcd(a,b)}$ or $lcm(a,b)\gcd(a,b) = ab$.

- For any positive integers a and b, $lcm(a,b) = ab$ if and only if $\gcd(a,b) = 1$.

Proof: By above theorem, $lcm(a,b)\gcd(a,b) = ab$.

If $\gcd(a,b) = 1$ if and only if $lcm(a,b) = ab$

Properties:

1. If $k > 0$, $lcm(ka,kb) = klcm(a,b)$

Proof: $\gcd(ka,kb) = k\gcd(a,b)$.

$$lcm(ka,kb)\gcd(ka,kb) = k^2ab = k\ lcm(a,b) \cdot k\gcd(a,b)$$

$\Rightarrow\ lcm(ka,kb) = k\ lcm(a,b)$.

2. If m is any common multiple of a and b, then $lcm(a,b)|m$.

Proof: Let $k = lcm(a,b)$. By division algorithm, $m = qk + r, 0 \leq r < k$.

i.e., $r = m - qk$.

Both a and b divide m and k implies a and b divide r.

Therefore, r is a common multiple of a and b.

But $0 \leq r < k \ \Rightarrow r = 0$.

Hence, $m = qk$ or $k|m$.

3. $\gcd(a,b) = lcm(a,b)$ if and only if $a = \pm b$

Proof: If $a = \pm b$ then $\gcd(a, b) = |a| = lcm(a, b)$.

Conversely, suppose $\gcd(a, b) = lcm(a, b) = k$

$a | lcm(a, b) = k$ and $k = \gcd(a, b) \, | a \Longrightarrow k = \pm a$

Similarly, $k = \pm b$.

Therefore, $a = \pm b$.