Quadrant II - Notes

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Unit: 5- Maxwell's Equations

Module Name: Generalization of Ampere's Law- Displacement current

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Notes:

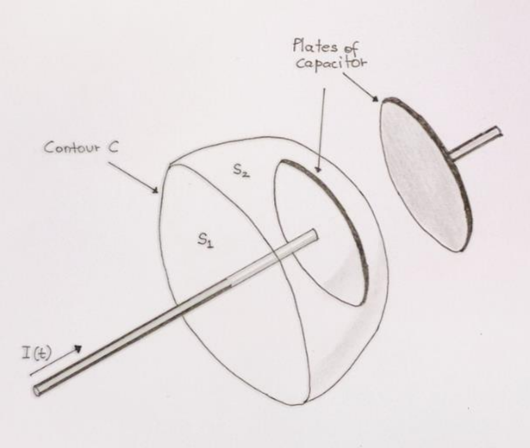
The generalization of Ampere's Law. Displacement Current

The magnetic field due to a current distribution satisfied Ampere's circuital law.

$$\oint \boldsymbol{H} \cdot \boldsymbol{dl} = \int_S \boldsymbol{J} \cdot \boldsymbol{n}\, \boldsymbol{da} \qquad - \mathrm{Eq}(1)$$

We shall now examine this law, show that it sometimes fails, and find a generalization that is always valid.

Consider the circuit shown in figure below, which consists of a small parallel plate capacitor being charged by a constant current ***I***



If Ampere’s law is applied to the contour $\boldsymbol{C}$ and the surface $\boldsymbol{S_1}$, we find

$$\oint \boldsymbol{H}\cdot \boldsymbol{dl} = \int_{\boldsymbol{S_1}} \boldsymbol{J}\cdot \boldsymbol{n}\, \boldsymbol{da} = \boldsymbol{I} \qquad -\text{Eq}(2)$$

If on the other hand, Ampere’s law is applied to the contour $\boldsymbol{C}$ and surface $\boldsymbol{S_2}$**,** then J is zero at all points on $\boldsymbol{S_2}$ and

$$\oint \boldsymbol{H}\cdot \boldsymbol{dl} = \int_{\boldsymbol{S_2}} \boldsymbol{J}\cdot \boldsymbol{n}\, \boldsymbol{da} = \boldsymbol{0} \qquad -\text{Eq}(3)$$

Equations (2) and (3) contradict each other and thus cannot both be correct. If $\boldsymbol{C}$ is imagined to be a great distance from the capacitor, it is clear that the situation is not substantially different from the standard Ampere’s law cases. One is thus led to think that $\text{Eq}(2)$ is correct, since it is not dependent of the new feature, namely the capacitor. $\text{Eq}(3)$, on the other hand, requires consideration of the capacitor for its deduction. It would appear, then, that $\text{Eq}(3)$ requires modification.

$$\boldsymbol{\nabla}\times \boldsymbol{H} = \boldsymbol{J} \qquad -\text{Eq}(4)$$

this also requires modification.

The proper modification can be made by nothing that $\text{Eq}(2)$ and $\text{Eq}(3)$ give different results because the integrals on the right-hand sides are different.

Phrased mathematically,

$$\int_{\boldsymbol{S_2}} \boldsymbol{J}\cdot \boldsymbol{n_2}\ \boldsymbol{da} - \int_{\boldsymbol{S_1}} \boldsymbol{J}\cdot \boldsymbol{n_1}\ \boldsymbol{da} \neq \boldsymbol{0} \qquad -\text{Eq}(5)$$

$\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ together form a closed surface (they join at $\boldsymbol{C}$); however, $\boldsymbol{n_2}$ is outward drawn and $\boldsymbol{n_1}$ inward drawn. If this fact is taken into account, $\mathrm{Eq}(5)$ may be written

$$\oint_{S_1+S_2} \boldsymbol{J}\cdot\boldsymbol{n}\,\boldsymbol{da} \neq \boldsymbol{0} \qquad - \mathrm{Eq}(6)$$

Phrased physically, the net transport current through the closed surface $\boldsymbol{S_1} + \boldsymbol{S_2}$ does not vanish because charge is piling up on the plate of the condenser enclosed by the surface. Charge conservation requires, according to (ENTER EQUATION HERE),

$$\oint_{S_1+S_2} \boldsymbol{J}\cdot\boldsymbol{n}\,\boldsymbol{da} = -\int_V \frac{\partial\boldsymbol{\rho}}{\partial\boldsymbol{t}}\,\boldsymbol{dv} \qquad - \mathrm{Eq}(7)$$

Because inside the volume $\boldsymbol{V}$ enclosed by $\boldsymbol{S_1} + \boldsymbol{S_2}$ the charge density $\boldsymbol{\rho}$ is changing with time on the condenser plate. In differential form $\mathrm{Eq}(7)$ is expressed by the equation of continuity,

$$\nabla\cdot\boldsymbol{J} + \frac{\partial\boldsymbol{\rho}}{\partial\boldsymbol{t}} = \boldsymbol{0} \qquad - \mathrm{Eq}(8)$$

It is now clear what is wrong with $\mathrm{Eq}(4)$: Taking its divergence we have

$$\nabla\cdot\nabla\times\boldsymbol{H} = \boldsymbol{0} = \nabla\cdot\boldsymbol{J},$$

since the divergence of a curl is identically zero. Thus the relation $\nabla\cdot\boldsymbol{J} = \boldsymbol{0}$, as implied by $\mathrm{Eq}(4)$, is consistent with charge conservation in the present situation, and so something must be added to the right side of $\mathrm{Eq}(4)$ that will give $\partial\boldsymbol{\rho}/\partial\boldsymbol{t}$ in $\mathrm{Eq}(8)$. What this could be is seen from the relation of $\boldsymbol{\rho}$ to electric displacement:

$$\nabla\cdot\boldsymbol{D} = \boldsymbol{\rho} \qquad - \mathrm{Eq}(9)$$

Inserting $\boldsymbol{\rho}$ from the $\mathrm{Eq}(9)$ into $\mathrm{Eq}(8)$, we get

$$\nabla\cdot\boldsymbol{J} + \frac{\partial}{\partial\boldsymbol{t}}\nabla\cdot\boldsymbol{D} = \nabla\cdot\left(\boldsymbol{J} + \frac{\partial\boldsymbol{D}}{\partial\boldsymbol{t}}\right) = \boldsymbol{0}.$$

If $\partial\boldsymbol{D}/\partial\boldsymbol{t}$ is added to $\mathrm{Eq}(4)$, then its divergence will correctly give $\mathrm{Eq}(8)$. (ENTER STATEMENT HERE). Inclusion of $\partial\boldsymbol{D}/\partial\boldsymbol{t}$ gives the generalized Ampere's law:

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial \boldsymbol{t}}. \qquad - \mathrm{Eq}(10)$$

The introduction of the second term on the right, which is known as the displacement current, represents one of Maxwell’s major contributions to electromagnetic theory.
