Quadrant II – Transcript and Related Materials

Programme : **Bachelor of Science (Third Year)**

Subject : **Physics**

Course Code : **PYC 110**

Course Title : **Electromagnetic Theory II & Theory of Relativity**

Unit 8 : **Relativistic Kinematics**

Module Name : **Derivation of the Lorentz Transformations and derivation of its consequences such as Length Contraction and Time dilation Part - I**

Module No : **07**

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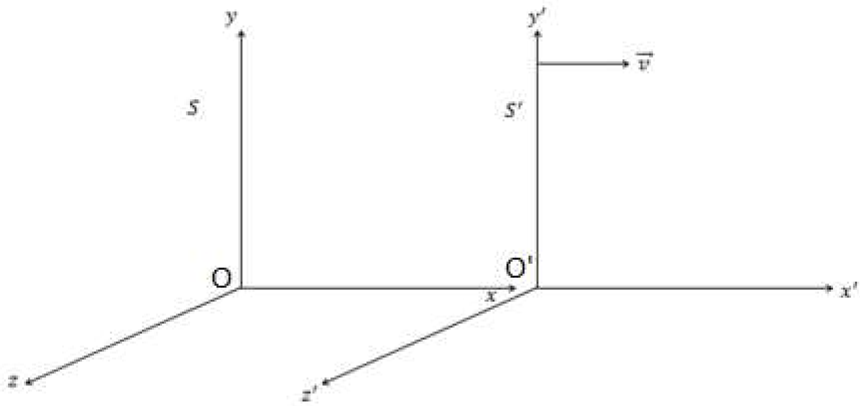
Notes

Derivation of the Lorentz Transformation Equations:



Consider two frames of reference S and S'.

S is fixed in space and S' is moving with uniform velocity v along x-axis with respect to S frame.

Consider an event in one inertial reference frame S, characterized by its location and time specified by the coordinates x, y, z, t

In the second inertial frame S', this same event is recorded as the space-time coordinates x', y' z' t'.

We seek the functional relationship: x' = x'(x,y,z,t), y' = y'(x,y,z,t), z' = z'(x,y,z,t) and t' = t'(x,y,z,t). That is, we want the equations of transformation which relate one observer's space-time coordinates of an event with the other observer's coordinates of the same event.

We shall use the fundamental postulates of special theory of relativity and the assumption that space and time are homogenous that is all points in space and time are equivalent.

Let S'-frame move with the relative velocity v along a common x-x' axis and by keeping corresponding planes parallel.

When origins O and O' coincide, let both clocks read t = 0 and t' = 0 respectively.

Homogeneity assumptions requires that transformation equations must be linear (i.e. they involve only the first power in the variables).

General form of transformation equations is given by:

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$

$$y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t$$

$$z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

In above equations subscripted coefficients are constants which must be determined to obtain the exact transformation equations. The values of these 16 coefficients depend upon the velocity v of the two inertial frames.

For example, if $v = 0$, then two frames coincide at all times and we expect $a_{11} = a_{22} = a_{33} = a_{44} = 1$, all other coefficients being zero.

If v is small compared to c, the coefficients should lead to the Galilean transformation equations. We seek to find the coefficients for any value of v, that is, as a function of v.

To determine the values of these 16 coefficients, we use the postulates of relativity, namely (1) The principle of relativity – no preferred inertial system exists, the laws of physics are same in all inertial systems and (2) The principle of constancy of the speed of light – the speed of light in free space has same value c in all inertial systems.

The x-axis coincides continuously with x'-axis. Therefore y = 0 and z = 0. It also follows that, y' = 0, z' = 0.

Hence, transformation formulas for y and z must be of the form:

$y' = a_{22}y + a_{23}z$

$z' = a_{32}y + a_{33}z$

That is, the coefficients, $a_{21,}\ a_{24,}\ a_{31,}\ a_{34}$ must be zero.

Likewise, the x-y plane (which is characterized by z = 0) should transform over to the x'-y' plane (which is characterized by z' = 0).

Similarly, for the x-z and x'-z' planes, y=0 should give y'=0.

Hence, it follows that a_{23} and a_{32} are zero so that; $y' = a_{22}y$ and $z' = a_{33}z$

These remaining constants coefficients a_{22} and a_{33}, can be evaluated using the relativity postulate. We illustrate for a_{22}. Suppose that we have a rod lying along the y-axis, measured by S to be of unit length.

According to the S'-observer, the rod's length will be a_{22} (i.e. $y' = a_{22} \times 1$).

Now, suppose that the very same rod is brought to rest along the y' axis of the S'-frame. The primed observer must measure the same length (unity) for this rod when it is at rest in his frame as the unprimed observer measures when the rod is at rest with respect to him; otherwise, there would be an asymmetry in the frames. In this case, however, the S-observer would measure the rod's length to be $1/a_{22}$ $\left[i.e.\, y = (1/a_{22})y' = (1/a_{22}) \times 1\right]$

Now, because of the reciprocal nature of these length measurements, the first postulates requires that these measurements are identical, otherwise the frames would not be equivalent physically.

Hence, we must have, $a_{22} = 1/a_{22}$ or $a_{22} = 1$.

The argument is identical in determining that $a_{33} = 1$.

Therefore, two transformation equations become:

$y' = y\ and\ z' = z - - - - - - - - - - (2)$

There remain transformation equations for x' and t', namely,

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t$$

Let us look at the t'-equation. For reasons of symmetry, we assume that t' does not depend on y and z. Otherwise, clocks placed symmetrically in the y-z plane (such as at +y, -y or +z, -z) about the x-axis would appear to disagree as observed from S', which would contradict the isotropy of space. Hence, $a_{42} = a_{43} = 0$. As for the x'-equation, we know that a point having x'=0 appears to move in the direction of the positive x-axis with the speed v, so that the statement x'=0 must be identical to the statement $x = vt$.

Therefore, we expect, $x' = a_{11}(x - vt)$ to be correct transformation equation. (That is, $x = vt$ always gives x'=0 in this equation.)

$$x' = a_{11}x - a_{11}vt = a_{11}x + a_{14}t$$

This gives, $a_{14} = -va_{11}$

Thus, our four equations have now been reduced to;

$$x' = a_{11}(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = a_{41}x + a_{44}t \text{ ----------- } (3)$$

There remains the task of determining the three coefficients $a_{11}, a_{41}\ and\ a_{44}$.

To do this, we use the principle of the constancy of the velocity of light. Let us assume that at the time t=0 a spherical electromagnetic wave leaves the origin of S, which coincides with the origin of S' at that moment. The wave propagates with a speed c in all directions in each inertial frame. Its progress, then, is described by the equation of sphere whose radius expands with time at a rate c in terms of either the primed or unprimed set of coordinates. That is,

$$x^2 + y^2 + z^2 = c^2t^2 \text{ ------------ } (4)$$

or, $x'^2 + y'^2 + z'^2 = c^2t'^2 \text{ ---------- } (5)$

Substituting transformation equations (3) in equation (5), we get,

$$a_{11}^2(x - vt)^{\ 2} + y^2 + z^2 = c^2(a_{41}x + a_{44}t)^2$$

Rearranging the terms gives as;

$$(a_{11}^2 - c^2a_{41}^2)x^2 + y^2 + z^2 - 2(va_{11}^2 + c^2a_{41}a_{44})xt = (c^2a_{44}^2 - v^2a_{11}^2)t^2$$

For this expression to agree equation (4), which represents the same thing, we must have,

$$c^2a_{44}^2 - v^2a_{11}^2 = c^2$$

$$a_{11}^2 - c^2a_{41}^2 = 1$$

$$va_{11}^2 + c^2a_{41}a_{44} = 0$$

Here we have three equations in three unknowns, whose solutions is given by;

$$a_{44} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$a_{11} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$a_{41} = \frac{-\frac{v}{c^2}}{\sqrt{1 - v^2/c^2}} - - - - - - - - - - - - (6)$$

By substituting these values into equations (3), we obtain Lorentz transformation equations as:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \left(\frac{v}{c^2}\right)x}{\sqrt{1 - v^2/c^2}} - - - - - - - - - - - (7)$$

For small velocities:

$$v << c$$

$\therefore \frac{v^2}{c^2}$ *becomes negligible in comparison with* 1. *Also* $\left(\frac{vx}{c^2}\right)$ *becomes negligible*.

Thus, Lorentz transformation equations reduces to Galilean transformation equations as follows;

$x' = x - vt$

$y' = y$

$z' = z$

$t' = t$