Hello students, welcome to this course Electromagnetic Theory II and Theory of Relativity. This is Part 2, Theory of Relativity. The name of this unit is Relativistic Dynamics, and the name of the module is Relativistic Force Law and Dynamics of a Single Particle. I'm Yatin Desai, Assistant Professor in Physics, Chowgule College, Margao.

Outline of this module is relativistic force law and dynamics of a single particle. Learning Outcomes: at the end of this module, you will be able to deduce relativistic force law and you will be able to obtain an expression for relativistic kinetic energy and relativistic total energy for a single particle in motion.

The Relativistic force law and the dynamics of a single particle: Newton's second law can be generalized to relativistic mechanics as given by this equation, $\vec{F} = \frac{d\vec{p}}{dt}$ and since p is mv, m we can replace by $\left(\frac{m_0\vec{v}}{\sqrt{1-v^2/c^2}}\right)$; we get that equation as; $\frac{d}{dt}\left(\frac{m_0\vec{v}}{\sqrt{1-v^2/c^2}}\right)$; we call this as equation #1. When F is 0, the quantity in the bracket is constant. That is, in absence of external forces, The momentum is conserved. If F defined by equation (1) is not zero, the total relativistic momentum changes by an amount delta P, then this change is equal to the total impulse $\int \vec{F}\,dt$ given to the system. Hence, the force defined by equation (1) has the general properties. We note that, the new form of Newton’s second law given by equation (1) is not equivalent to writing $\vec{F} = m\vec{a}$ which is equal to $\frac{m_0\vec{v}}{\sqrt{1-v^2/c^2}}\left(\frac{d\vec{v}}{dt}\right)$. In Newtonian mechanics, we defined the kinetic energy, K, of a particle to be equal to the work done by an external force in increasing the speed of the particle from zero to some value v. That is: $K = \int_{v=0}^{v=v} \vec{F}.\vec{dl}$; where $\vec{F}.\vec{dl}$ is the work done by the force $\vec{F}$ in displacing the particle through $\vec{dl}$. Writing equation (2) for a case of motion in one dimension, say x; Classically, $K = \int_{v=0}^{v=v} F\,dx = \int m_0\left(\frac{dv}{dt}\right)dx$, i.e. $K = \int m_0 dv$ and we write that dt below dx so it is $\frac{dx}{dt}$ and in the next equation, we write that $\frac{dx}{dt}$ as v and after integrating, we get, $K = \frac{1}{2}m_0v^2$. Here, we have written the mass of the particle as m_0 to emphasize that, in Newtonian mechanics, we do not regard the mass as varying with the speed, and we take the force to be $m_0a = m_0\, {du}/{dt}$. In relativistic mechanics, it proves useful to use a corresponding definition for kinetic energy in which we use relativistic equation of motion given by equation (1), rather than the Newtonian one.

Thus, relativistically, we can write $K = \int_{v=0}^{v=v} F\,dx$, we get $K = \int \frac{d}{dt}(mv)dx \therefore K = \int d(mv)\frac{dx}{dt}$ and after differentiating this (mv), it is (mdv+vdm) because m also varies with the motion. $\therefore K = \int_{v=0}^{v=v}$ we have $(mvdv + v^2dm)$ after multiplying this v inside this bracket. This is equation number (3). In equation (3) we notice that, m and v both are variables. These quantities are related by the equation $m = \frac{m_0}{\sqrt{1-v^2/c^2}}$ which can be rewritten as; $m^2c^2 - m^2v^2 = m_0^2c^2$. This is written by squaring this equation on both the sides and then cross multiplying by making this common denominator. Now, after having written this equation, we take the differential of this equation on both the sides which gives us, $2mc^2dm - m^2 2vdv - v^2 2mdm = 0$. Because this quantity on the right-hand side is constant. On cancelling 2M, or on dividing this equation by 2m we will get and then multiplying it with -1, we get $\mathrm{mvdv} + \mathrm{v}^2\mathrm{dm}$ and this will be minus sign which is taken on the other side which will be $\mathrm{c}^2\mathrm{dm}$. That means we have divided throughout this equation by -2m and we get this equation as $mvdv + v^2dm = c^2dm$. The left-hand side of this equation is exactly equal to the integrand of equation #3. Let us see this left-hand side is exactly equal to the integrand of this equation #3. That means, in this place we can substitute with c square dm and that is what we can do in the next step. Hence, we can write the relativistic expression for the kinetic energy of a particle as $\mathrm{K} = \int_{\mathrm{v}=0}^{\mathrm{v}=\mathrm{v}} \mathrm{c}^2\mathrm{dm}$. Since c^2 is constant, it is taken out of this integral and it is ;

$c^2 \int_{v=0}^{v=v} dm$; an integral of differentiation of m is m itself and the limits are from m to m_0; (m is the mass), m_0 is the mass (at) when velocity is zero which is rest mass, m is the mass when the particle is having the velocity v. So, $mc^2 - m_0c^2$ which we call as equation #4. By using equation $m = \frac{m_0}{\sqrt{1-v^2/c^2}}$; we get; K *as* m_0 *divided by* $\sqrt{1-v^2/c^2}$ *times* c^2 *minus* m_0. So we can take that m_0c^2 *as common and write inside the bracket as only* $\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)$, we call this equation #5. Since the total energy of the particle $E = mc^2$, we write equation (4) as; $E = m_0c^2 + K$; where, m_0c^2 is the rest energy of the particle which is the energy of the particle at rest, when $v = 0$ and $K = 0$. The total energy of the particle is the sum of its rest energy and its kinetic energy.

To show that the relativistic expression for K reduces to the classical result when $v \ll c$, consider equation (5) given by; $K = m_0c^2\left(1/\sqrt{1-v^2/c^2} - 1\right)$ i.e. $K = m_0c^2\left[\left(1-v^2/c^2\right)^{-1/2} - 1\right]$. And, using the binomial expression, we get expansion, we get, $K = m_0c^2\left[1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \cdots \ldots \ldots - 1\right]$ *and this becomes* $\frac{1}{2}m_0v^2$, and we have ignored the higher order terms. As $v \to c$ in equation (5), the kinetic energy K tends to infinity. That is from equation (4), an infinite amount of work would need to be done on the particle to accelerate it up to the speed of light. Thus, again, c plays the role of a limiting velocity. It is also noted from equation #(4); $K = (m - m_0)c^2$, that a change in the kinetic energy of a particle is related to a change in its inertial mass. To obtain the connection between the kinetic energy of a rapidly moving particle and its momentum p. This is done by eliminating v between equation (5) and the equation $\vec{p} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\vec{v}$; we get, $(K + m_0c^2)^2 = (pc)^2 + (m_0c^2)^2$equation #(7). Since the total energy is given as; $E = K + m_0c^2$; we get; $E^2 = (pc)^2 + (m_0c^2)^2$. This is equation #8. A mnemonic device or a memory device to remember these equations (7) and (8) is shown in the figure below, which is the right-angled triangle; this side is m_0c^2. This side is pc. This full side which is E which is mc^2, and it is a combination of $m_0c^2 + K^2$ or E equals $(m_0c^2 + K)$ or E^2equals $(m_0c^2)^2 + (pc)^2$.

These are the references for this module.

Thank you.