Quadrant II – Transcript and Related Materials

Programme : **T. Y. B. Sc.**

Subject : **Mathematics**

Course Code : **MTC-108**

Course Title : **Differential Equation-II**

Unit : **II- Power Series Solution of Some Linear Equations**

Module Name : **Generating Function (Part2)**

Name of the Presenter : **Sujata Desai**

Notes

Bessel Equation

If α is a constant, $\mathrm{Re}\,\alpha \geq 0$, the *Bessel equation of order* α is the equation

$$x^2y" + xy' + (x^2 - \alpha^2)y = 0.$$

This has the form

$$x^2y" + xa(x)y' + b(x)y = 0,$$

with

$$a(x) = 1, \qquad b(x) = x^2 - \alpha^2$$

Since $a,\ b$ are analytic at $x = 0$, the Bessel equation has the origin as a regular singular point.
The indicial polynomial q is given by

$$q(r) = r(r-1) + r - \alpha^2 = r^2 - \alpha^2,$$

Whose two roots $r_1,\ r_2$ are

$$r_1 = \alpha, \qquad r_2 = -\alpha$$

- Bessel Function of zero order of the first kind

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}$$

- Bessel Function of zero order of the second kind

$$K_0(x) = -\sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right)\left(\frac{x}{2}\right)^{2m} + (\log x)J_0(x).$$

- Bessel Function of order α of the first kind

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m}, \quad (Re\,\alpha \geq 0).$$

- Bessel Function of order n of the second kind

$$K_n(x) = -\frac{1}{2}\left(\frac{x}{2}\right)^{-n} \sum_{j=1}^{n-1} \frac{(n-j-1)!}{j!}\left(\frac{x}{2}\right)^{2j} - \frac{1}{2}\frac{1}{n!}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\left(\frac{x}{2}\right)^n$$
$$-\frac{1}{2}\left(\frac{x}{2}\right)^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,(m+n)!}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)+\left(1+\frac{1}{2}+\cdots+\frac{1}{m+n}\right)\right]\left(\frac{x}{2}\right)^{2m}$$
$$+(\log x)J_n(x)$$

Generating Function for Bessel Function $J_n(x)$

$$e^{\left\{\frac{1}{2}x\left(z-\frac{1}{z}\right)\right\}} = \sum_{n=-\infty}^{\infty} J_n(x) z^n$$

- when n is a positive integer, $J_n(x)$ is the coefficient of z^n in the expression of $e^{\left\{\frac{1}{2}x\left(z-\frac{1}{z}\right)\right\}}$ in ascending and descending power of z.
- J_n is the coefficient of z^{-n} multiplied by $(-1)^n$ in the expansion of the above expression.

  $e^{\left\{\frac{1}{2}x\left(z-\frac{1}{z}\right)\right\}}$ is called the Generating Function of Bessel Function $J_n(x)$

Proof: We have

$$e^{\left\{\frac{1}{2}x\left(z-\frac{1}{z}\right)\right\}} = e^{\frac{xz}{2}-\frac{x}{2z}} = e^{\frac{xz}{2}}.e^{-\frac{x}{2z}}$$
$$= \left[1+\left(\frac{xz}{2}\right)+\left(\frac{xz}{2}\right)^2\frac{1}{2!}+\left(\frac{xz}{2}\right)^3\frac{1}{3!}+\cdots+\left(\frac{xz}{2}\right)^n\frac{1}{n!}+\left(\frac{xz}{2}\right)^{n+1}\frac{1}{(n+1)!}+\cdots\right]$$
$$\times\left[1+\left(\frac{-x}{2z}\right)+\left(\frac{-x}{2z}\right)^2\frac{1}{2!}+\left(\frac{-x}{2z}\right)^3\frac{1}{3!}+\cdots+\left(\frac{-x}{2z}\right)^n\frac{1}{n!}+\left(\frac{-x}{2z}\right)^{n+1}\frac{1}{(n+1)!}+\cdots\right]$$
$$= \left[1+\left(\frac{x}{2}\right)z+\left(\frac{x}{2}\right)^2\frac{z^2}{2!}+\left(\frac{x}{2}\right)^3\frac{z^3}{3!}+\cdots+\left(\frac{x}{2}\right)^n\frac{z^n}{n!}+\left(\frac{x}{2}\right)^{n+1}\frac{z^{n+1}}{(n+1)!}+\cdots\right]$$
$$\times\left[1+(-1)\left(\frac{x}{2}\right)z^{-1}+(-1)^2\left(\frac{x}{2}\right)^2\frac{z^{-2}}{2!}+(-1)^3\left(\frac{x}{2}\right)^3\frac{z^{-3}}{3!}+\cdots+(-1)^n\left(\frac{x}{2}\right)^n\frac{z^{-n}}{n!}\right.$$
$$\left.+(-1)^{n+1}\left(\frac{x}{2}\right)^{n+1}\frac{z^{-(n+1)}}{(n+1)!}+\cdots\right]$$

Now the coefficient of z^n in the product above is obtained by multiplying the coefficient of $z^n, z^{n+1}, z^{n+2}, \cdots$ in the first bracket with the coefficient of $z^0, z^{-1}, z^{-2}, \cdots$ in the second bracket respectively and thus

$$\text{Coefficient of } z^n = \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!}\frac{1}{(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \cdots$$
$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

$$= \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r}$$

$$= J_n(x)$$

Similarly the coefficient of z^{-n} in the product above is obtained by multiplying the coefficient of $z^{-n}, z^{-(n+1)}, z^{-(n+2)}, \cdots$ in the second bracket with the coefficient of $z^0, z^1, z^2, \cdots$ in the first bracket respectively and thus

$$\text{Coefficient of } z^{-n} = \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!}\frac{(-1)^{n+2}}{(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \cdots$$

$$= \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n - \frac{(-1)^n}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!}\frac{(-1)^n}{(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \cdots$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r(-1)^n}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

$$= (-1)^n \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r}$$

$$= (-1)^n J_n(x) = J_{-n}(x)$$

Finally, the coefficient of z^0 is obtained by multiplying the coefficient of $z^0, z^1, z^2, \cdots$ in the first bracket with the coefficient of $z^0, z^{-1}, z^{-2}, \cdots$ in the second bracket as

$$1 - \left(\frac{x}{2}\right)^2 + \left(\frac{x}{4}\right)^4 \left(\frac{1}{2!}\right)^2 - \left(\frac{x}{2}\right)^6 \left(\frac{1}{3!}\right)^2 + \cdots = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2m} = J_0(x)$$

Thus

$$e^{\left\{\frac{1}{2}x\left(z-\frac{1}{z}\right)\right\}} = J_0(x)z^0 + \left(z - \frac{1}{z}\right)J_1(x) + \left(z^2 + \frac{1}{z^2}\right)J_2(x) + \cdots + \left(z^n + (-1)^n\frac{1}{z^n}\right)J_n(x)$$

$$= \sum_{n=-\infty}^{\infty} z^n J_n(x)$$