Quadrant II – Notes

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Course Title: Metric Spaces

Unit: Introductory Concepts in Metric Spaces

Module Name: Limit Points and Isolated Points I

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METRIC SPACES

MTC 110

Module 8: Limit Points and Isolated points – I

Defn. (Limit point of a set)

Let (X, d) be a metric space. Let $A \subseteq X$. A point $x_0 \in X$ is called a limit point of A if $\forall r > 0$, $(B(x_0, r) \cap A) \setminus \{x_0\} \neq \phi$.

Defn (Isolated point of a set)

Let (X, d) be a metric space. Let $A \subseteq X$. A point $x_0 \in A$ is called an isolated point of A if $\exists r > 0 \ni B(x_0, r) \cap A \subseteq \{x_0\}$

Example 1 Show that every point in $\mathbb{R}$ is a limit point of $\mathbb{Q}$.

Soln. For any $x \in \mathbb{R}$ and for any $r > 0$

$$B(x, r) = (x - r, x + r)$$

$$\text{and} \left((x - r, x + r) \cap \mathbb{Q} \right) \setminus \{x\} \neq \phi.$$

$\therefore$ x is a limit point of $\mathbb{Q}$.

Example 2 Identify all limit points of $A = (0, 1)$ in $(\mathbb{R}, d)$ where $d =$ usual metric.

Soln: Case I Let $x > 1$

Then $(1, \infty)$ is an open set containing x.

$\therefore \exists r > 0 \ni B(x, r) \subseteq (1, \infty)$ and

$$\left(B(x, r) \cap (0, 1) \right) \setminus \{x\} \subseteq \left((1, \infty) \cap (0, 1) \right) \setminus \{x\} = \phi$$

$$\therefore \left(B(x, r) \cap (0, 1) \right) \setminus \{x\} = \phi$$

$\therefore$ x is not a limit point of A.

Case 2: Let $x < 0$

Then $(-\infty, 0)$ is an open set containing x

$\therefore \exists\, r > 0 \ni B(x,r) \subseteq (-\infty, 0)$

$\therefore \left(B(x,r) \cap (0,1)\right) \setminus \{x\} \subseteq \left((-\infty,0) \cap (0,1)\right) \setminus \{x\}$
$= \phi$

$\therefore \left(B(x,r) \cap (0,1)\right) \setminus \{x\} = \phi$

$\therefore$ x is not a limit point of A.

Case 3 Let $x = 0$

$\forall r > 0$, $B(0,r) = (-r, r)$

and $\left(B(0,r) \cap A\right) \setminus \{0\} \neq \phi$

$\therefore$ $x = 0$ is a limit point of A.

Case 4 Let $x = 1$

$\forall r > 0$, $B(1,r) = (1-r, 1+r)$

and $\left(B(1,r) \cap A\right) \setminus \{1\} \neq \phi$

$\therefore$ $x = 1$ is a limit point of A.

Case 5 Let $0 < x < 1$

$\forall r > 0$, $B(x,r) = (x-r, x+r)$

and $\left(B(x,r) \cap A\right) \setminus \{x\} \neq \phi$

$\therefore$ x is a limit point of A.

$\therefore$ The set of all limit points of A is $[0,1]$

Exercise In $(\mathbb{R}, d)$, d = usual metric, show that $\mathbb{N}$ has no limit points.

Solution Case 1 Let $x_0 \notin \mathbb{N}$ and $x_0 > 1$

$\Rightarrow x_0 \in (n, n+1)$

Since $(n, n+1)$ is an open set, $\exists\, r > 0 \ni B(x_0, r) \subseteq (n, n+1)$

$$\therefore \left(B(x_0, r) \cap \mathbb{N}\right) \setminus \{x_0\} \subseteq \left((n, n+1) \cap \mathbb{N}\right) \setminus \{x_0\}$$
$$= \phi$$

$$\therefore \left(B(x_0, r) \cap \mathbb{N}\right) \setminus \{x_0\} = \phi$$

$\therefore x_0$ is not a limit point of $\mathbb{N}$.

Case 2 Let $x_0 < 1$

$\Rightarrow x_0 \in (-\infty, 1)$

$\Rightarrow \exists\, r > 0 \ni B(x_0, r) \subseteq (-\infty, 1)$

Now, $\left(B(x_0, r) \cap \mathbb{N}\right) \setminus \{x_0\} \subseteq \left((-\infty, 1) \cap \mathbb{N}\right) \setminus \{x_0\} = \phi$

$$\therefore \left(B(x_0, r) \cap \mathbb{N}\right) \setminus \{x_0\} = \phi$$

$\therefore x_0$ is not a limit point of $\mathbb{N}$.

Case 3 If $x_0 \in \mathbb{N}$. n

$\exists\, r = \frac{1}{2} \ni$

$$B(x_0, \tfrac{1}{2}) = (x_0 - \tfrac{1}{2}, x_0 + \tfrac{1}{2})$$

and $\left(B(x_0, \frac{1}{2}) \cap \mathbb{N}\right) \setminus \{x_0\} = \left((x_0 - \frac{1}{2}, x_0 + \frac{1}{2}) \cap \mathbb{N}\right) \setminus \{x_0\}$

$$= \phi$$

$$\therefore \left(B(x_0, \tfrac{1}{2}) \cap \mathbb{N}\right) \setminus \{x_0\} = \phi$$

$\therefore x_0$ is not a limit point of $\mathbb{N}$.

$\therefore \mathbb{N}$ has no limit points in $\mathbb{R}$.

Exercise Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Find the limit points of S in $(\mathbb{R}, d)$, d = usual metric

Solution We will show that '0' is the only limit point of S.

Case 1 $p = 0$

$\forall r > 0$, $B(0, r) = (-r, r)$.

By Archimedean property $\exists N \in \mathbb{N} \ni \frac{1}{N} < r$

$\Rightarrow -r < \frac{1}{N} < r \Rightarrow \frac{1}{N} \in (-r, r)$

$\therefore (B(0, r) \cap S) \setminus \{0\} \neq \phi$.

$\therefore p = 0$ is a limit point of S.

Case 2 Let $p < 0 \Rightarrow p \in (-\infty, 0)$

$\Rightarrow \exists$ open ball $B(p, r) \subseteq (-\infty, 0)$

Then $(B(p, r) \cap S) \setminus \{p\} = \phi$.

$\therefore p < 0$ is not a limit point of S

Case 3 Let $p > 1 \Rightarrow p \in (1, \infty)$

$\Rightarrow \exists$ open ball $B(p, r) \subseteq (1, \infty)$

Then $(B(p, r) \cap S) \setminus \{p\} = \phi$

$\therefore p > 1$ is not a limit point of S.

Case 4 Let $p = 1$

$\exists r = \frac{1}{2} \ni B(p, r) = (\frac{1}{2}, \frac{3}{2})$

and $(B(p, r) \cap S) \setminus \{p\} = (B(1, \frac{1}{2}) \cap S) \setminus \{1\} = \phi$

$\therefore p = 1$ is not a limit point of S.

Case 5 Let $0 < p < 1 \ni p \notin S$. Then $\frac{1}{p} > 0$ & $\exists! m \in \mathbb{N} \ni m < \frac{1}{p} < m+1$

i.e. $\frac{1}{m+1} < p < \frac{1}{m} \Rightarrow \exists B(p, r) \subseteq (\frac{1}{m+1}, \frac{1}{m})$ and $(B(p, r) \cap S) \setminus \{p\} = \phi$

$\therefore p$ is not a limit point of S.

Case 6 $p \neq 1$ & $p \in S \Rightarrow p = \frac{1}{m}$ for some $m \in \mathbb{N}$, $m \neq 1$

$\Rightarrow p \in (\frac{1}{m+1}, \frac{1}{m-1}) \therefore \exists B(p, r) \subseteq (\frac{1}{m+1}, \frac{1}{m-1})$ and

$(B(p, r) \cap S) \setminus \{p\} = \phi$.

$\therefore p$ is not a limit point of S.