Quadrant II – Notes

Paper Code: CAC102

Module Name: Graphical Solutions of Linear Programming Problems with two variables only: Maximization of the objective function subject to the constraints

Linear Programming:

Linear programming is a technique for determining an optimum schedule of interdependent activities in view of the available resources. (Swarup, Gupta & Mohan, 2014, p. 39)

Basically a person in business is interested in maximizing profit and minimizing cost in view of the available resources. Linear programming technique helps him/her in determining an optimum schedule of interdependent activities in view of the available resources.

Components of Linear Programming problem(LPP):

- Decision variables
  (Decisions variables must always be non-negative)
- Objective function
- Constraints

Any LPP aims to minimize or maximize the given objective function subject to the constraints.

Decision variables:
Are the quantities that can be controlled.
They are always non- negative. In any Linear Programming Problem, our aim is to find the values of these variables so that we get optimum(i.e. minimum/maximum) value of the objective function.

Objective function:
A linear function of decision variables that we wish to minimize or maximize is called objective function. It is expressed as $z=ax+by$ (where a and b are constants & x and y are decision variables).

Constraints:
The restrictions or the limitations on the given decision variables are called as constraints. They can be linear inequalities or linear equations in terms of decision variables.

Some important definitions required for graphical solutions of Linear Programming Problem:

Feasible Region:

A region common to all the constraints and which also satisfies the non-negative restrictions on the decision variables is called **feasible region**.

If for a given LPP feasible region doesn't exists, then the solution also doesn't exists for the given LPP.

If feasible region exists it can be **bounded** or **unbounded**.

Feasible region is **bounded** if it can be enclosed within a circle.

Feasible region is **unbounded** if it cannot be enclosed within a circle.

Optimal Solution:

Any point in the feasible region which optimizes (i.e. minimizes or maximizes) the given objective function is called **optimal solution**.

Steps involved in solving a Linear Programming problem using Graphical method:

1) Represent each constraint on the graph.
2) Identify the feasible region.
3) Determine all the corner points(vertices) of the feasible region.
4) Find the value of the objective function at each of these corner points.
5) At this step we determine optimal solution.

   Case I:

   If the feasible region is bounded, then the objective function attains its maximum as well as minimum at, at least one of these corner points. So in this case we choose the corner point which optimizes (i.e. minimizes/maximizes) the value of the objective function. This point gives an optimal solution.

   Case II:

   If the feasible region is unbounded , then optimal solution may or may not exists, depending upon the specifics of the objective function. However if optimal solution exists, then it must occur at, at least one of these corner points of the feasible region.

Please note:

If the objective function attains its maximum at two adjacent corner points of the feasible region, then any point on the line segment joining these two points is also an optimal solution. So in such a case the LPP has infinitely many solutions. Same result is also true for minimization.

Problem:

Solve graphically:
Maximize z= 6x+8y
Subject to:

$$5x + 4y \leq 40$$
$$x + y \leq 9$$
$$x \geq 0 \text{ and } y \geq 0$$

Solution:

z= 6x+8y → Objective function

$5x + 4y \leq 40$, $x + y \leq 9$ → Constraints

$x \geq 0$ and $y \geq 0$ → Non-negative restrictions

Where x and y are decision variables

Step 1: Represent each constraint on the graph.

Since in any Linear Programming problem , $x \geq 0$ and $y \geq 0$, graphical solution of the problem is always confined to the first quadrant. So we restrict to the first quadrant.

To represent the constraint $5x + 4y \leq 40$, we first consider the equation $5x + 4y = 40$.
By substituting x=0 in $5x + 4y = 40$, we get,

$$5 \times 0 + 4y = 40$$
$$\therefore \quad 0 + 4y = 40$$
$$\therefore \quad 4y = 40$$
$$\therefore \quad y = 10$$

Let us call this point as A(0,10).

By substituting y=0 in $5x + 4y = 40$, we get,

$$5x + 4 \times 0 = 40$$
$$\therefore \quad 5x + 0 = 40$$
$$\therefore \quad 5x = 40$$
$$\therefore \quad x = 8$$

Let us call this point as B(8,0).

To represent the constraint $x + y \leq 9$, we first consider the equation $x + y = 9$.
By substituting x=0 in $x + y = 9$, we get,

$$0 + y = 9$$
$$\therefore \quad y = 9$$

Let us call this point as C(0,9).

By substituting y=0 in $x + y = 9$ we get,

$x + 0 = 9$

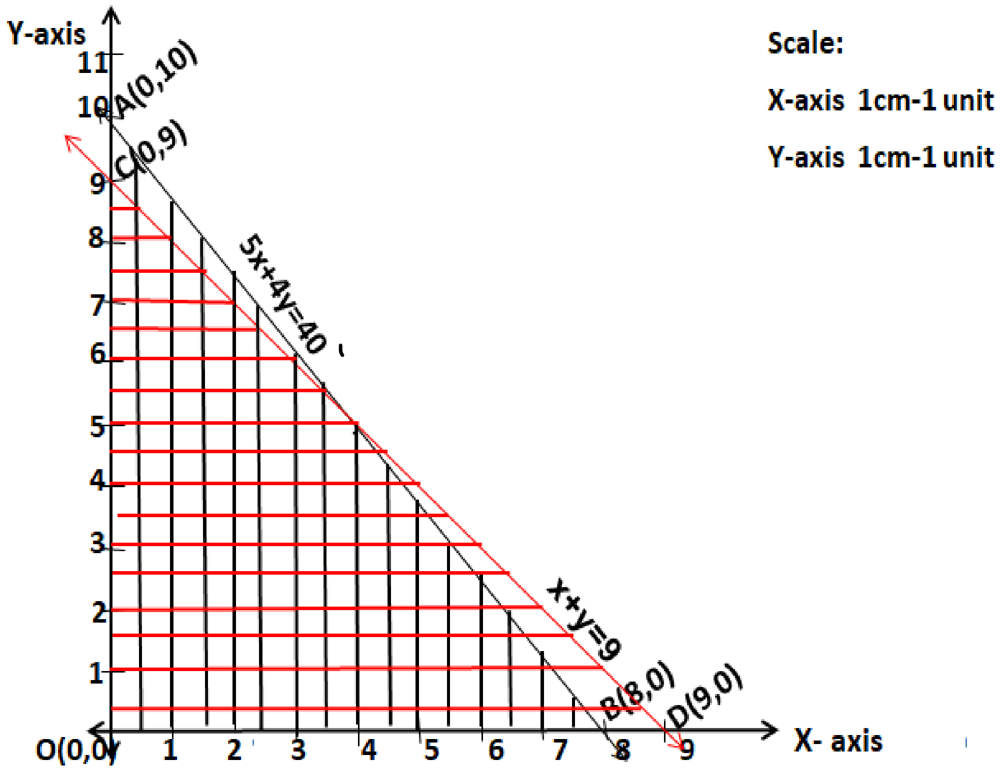
$\therefore \quad x = 9$

Let us call this point as D(9,0).

After this, the points A(0,10) , B(8,0), C(0,9) and D(9,0) are plotted on a XY plane.

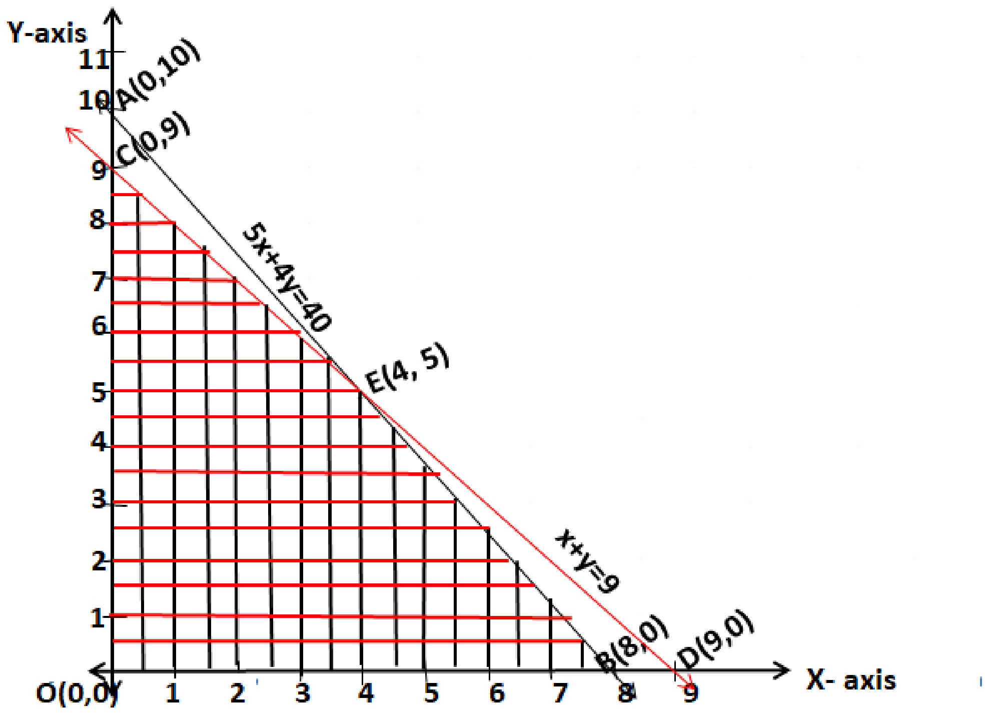
Next join the points A(0,10) and B(8, 0) to obtain the line 5x+4y=40.
Also, join the points C(0, 9) and D(9, 0) to obtain the line x+y=9.

The line 5x+4y=40 and the origin side of the line 5x+4y=40 together represent the constraint $5x + 4y \leq 40$. So starting from the line 5x+4y=40, we shade towards the origin (Refer graph no. 1).
The line $x + y = 9$ and the origin side of the line x+y=9 together represent the constraint $x + y \leq 9$. So starting from the line $x + y = 9$, we shade towards the origin (Refer graph no. 1).
For both the constraints we restrict the shading to the first quadrant because of non-negative restrictions on the decision variables.



Graph No. 1

Step 2: Identify the feasible region.
CEBO is the region common to both the constraints and it also satisfies the non-negative restrictions(refer graph no. 2) on the given decision variables. Therefore CEBO is the feasible region. It is polygonal region and hence bounded.



Graph No. 2

Step 3: Determine all the corner points of the feasible region.

C, E, B and O are the corner points of this feasible region. Coordinates of point C, B and O are already known. Please note that the point E is, the point of intersection of the lines 5x+4y=40 and x+y=9. Which can be obtained either by mere inspection or by solving the two equations 5x+4y=40 and x+y=9 simultaneously by using elimination method or Cramer's rule. So after solving both the equations simultaneously we get the point E as (4,5).

Step 4: Find the value of the objective function at each of these corner points.

Table No. 1

Corner points	$\mathbf{z = 6x + 8y}$
C(0,9)	$6 \times 0 + 8 \times 9 = 0 + 72 = 72$
E(4, 5)	$6 \times 4 + 8 \times 5 = 24 + 40 = 64$
B(8, 0)	$6 \times 8 + 8 \times 0 = 48 + 0 = 48$
O(0,0)	$6 \times 0 + 8 \times 0 = 0 + 0 = 0$

Step 5:

Since the feasible region is bounded, the objective function will attain its maximum at, at least one of these corner points. So we select the corner point at which z is maximum.
From table no. 1, we observe that maximum value of z is 72 which is attained at C(0, 9).