Hello students, welcome to this course, Classical Mechanics and Thermal Physics. This is Section I Classical Mechanics. This is Unit 4 Rigid Bodies. The name of the module is Euler’s equations of motion of a rigid body. I am Yatin Desai, Assistant Professor in Physics, Parvatibai Chowgule College, Margao-Goa.

Outline of this module is motion of a rigid body in space, equations of motion for a rigid body and conservation of angular momentum and kinetic energy in a torque free motion. Learning Outcomes: at the end of this module, you will be able to derive an expression for Euler's equation of motion for a rigid body, prove the conservation of kinetic energy for the torque free motion and prove, the conservation of angular momentum for the torque free motion.

Motion of a rigid body in space: Motion of a rigid body in space is described by the sets of all these 4 equations. This part describes the rectilinear motion or the linear motion, and this part describes the rotational motion. The basic equation of motion is $\frac{d\vec{p}}{dt} = \vec{F}$; force which is equation number (1) and for rotational motion, it is $\frac{d\vec{L}}{dt} = \vec{N}$; given by equation #2. Here, the linear momentum $\vec{p}$ is m times $\vec{v}$ which is equation #3, whereas, here in equation (2); the $\vec{L}$ the angular momentum is $\overleftrightarrow{I}.\vec{\omega}$; where $\overleftrightarrow{I}$ is the moment of inertia tensor. This is equation #4. In equation #3; $\vec{p}$ is always parallel to $\vec{v}$. Whereas in equation #4 $\vec{L}$ is not in general parallel to $\vec{\omega}$. In equation #3, m is the scalar and in equation #4, $\overleftrightarrow{I}$ is a tensor. In equation, #3, m is a constant for non-relativistic motion. Whereas in equation #4, $\overleftrightarrow{I}$ is not constant but it depends on the axis chosen in space. So, please note that in equation #4, $\overleftrightarrow{I}$ changes at the body moves. Next point is (no) there is no proper symmetrical set of axes that can be found so as to describe the orientation in space. Ok, all these things imply that $\overleftrightarrow{I}$ changes as body rotates

Euler’s Equations of Motion for a Rigid Body: Equation (2) which we have written in the previous slide, as $\vec{L}$ equal to the dot product between inertia tensor $\overleftrightarrow{I}$ and $\vec{\omega}$ It is the motion of the body with respect to axis fixed in space, and we also pointed out that moment of inertia $\overleftrightarrow{I}$, the inertia tensor $\overleftrightarrow{I}$ changes at the body rotates. So, to avoid the difficulty that $\overleftrightarrow{I}$ changes at the body rotates, we select the set of axes fixed in rotating body itself. Therefore, we use the relation for time derivative in two coordinates (systems), say in the fixed coordinate system and the moving coordinate system which is given by time derivative in the S-frame is $\frac{d}{dt}$ and time derivative in the S’-frame is $\frac{d^*}{dt} + \vec{\omega}\times$. It’s an operator equation for the time derivative. $\frac{d}{dt}$ is the time derivative in the S-frame which is fixed in space. $\frac{d^*}{dt}$ is the time derivative in S*-frame which is rotating with respect to the S-frame with an (angular momentum) angular velocity $\vec{\omega}$. Therefore, equation (2) can be written with respect to the axis in the moving coordinate system. Since the axis is to be fixed to the body as $\frac{d\vec{L}}{dt} = \frac{d^*\vec{L}}{dt} + \vec{\omega}\times\vec{L}$. We had $\frac{d\vec{L}}{dt} = \vec{N}$, but we changed that coordinate axis, from S to S* frame. Fix that axis to the rotating body itself in the S* frame and write that expression as not $\frac{d\vec{L}}{dt} = \vec{N}$ but $\frac{d^*\vec{L}}{dt} + \vec{\omega}\times\vec{L} = \vec{N}$. This is equation #5. So, $\overleftrightarrow{I}$ was not constant in S-frame, but now since we have fixed the chosen to frame as S*, $\overleftrightarrow{I}$ is constant relative to the body axis. Therefore, using equation #4 *i.e.* $\vec{L} = \overleftrightarrow{I}.\vec{\omega}$ in equation (5); we have $\frac{d^*\vec{L}}{dt}$ which is $I\vec{\omega} + \vec{\omega}\times\vec{L}$ which is $(I.\vec{\omega})$ equal to $\vec{N}$. Since in this time derivative I is constant, the way we have chosen our frame S*. So, I can be brought out of this operator $\frac{d^*}{dt}$. Therefore, $I\frac{d^*\vec{\omega}}{dt} + \vec{\omega}\times\left(\overleftrightarrow{I}.\vec{\omega}\right) = \vec{N}$. We have also, one relationship between the (angular velocity) time derivative of the angular velocity. That is $\frac{d^*\vec{\omega}}{dt} = \frac{d\vec{\omega}}{dt}$. Therefore, we can write this equation as $I\frac{d\vec{\omega}}{dt} + \vec{\omega} \times (I.\vec{\omega}) = \vec{N}$. Or the same equation is written as $I\dot{\omega}$, the 1st order time derivative of $\vec{\omega}$ is $\dot{\omega}$ plus $\vec{\omega} \times (I.\vec{\omega}) = \vec{N}$. This equation #6.

To write equation (6) in component form, consider this second term on the (right) left hand side. This is $\vec{\omega} \times (I.\vec{\omega})$. So. we'll only consider this term. $\vec{\omega} \times (I.\vec{\omega})$; so this cross product, we are writing by using a determinant. The first row elements are the unit vectors; $\hat{\imath}, \hat{\jmath}, \hat{k}$. The 2nd row elements are the components the first vector here, $\vec{\omega}$; which is ω_x, ω_y, and ω_z. And the third-row elements are the components of this $(I.\vec{\omega})$ which is $I_x\omega_x$, $I_y\omega_y$, $I_z\omega_z$. Therefore, $\vec{\omega} \times (I.\vec{\omega})$, when you solve this determinant, we get it as $\omega_y\ \omega_z$ which is common here as well as here and, in the bracket, we can write $\left(I_z - I_y\right)\hat{\imath}$ minus $\omega_x\ \omega_z$; it is common between these two products and these two products, we can write it outside the bracket and we can write it as $(I_z - I_x)\hat{\jmath}$ plus $\omega_x\ \omega_y$ which is appearing in this and inside the bracket we can write $\left(I_y - I_x\right)\hat{k}$. Therefore, we can write equation (6) in component form as the left-hand side. The first term on the left-hand side is $I\dot{\omega}$, the second term is here and right-hand side it is $\overrightarrow{N}$. The first term on the left-hand side can be written as $I_x\dot{\omega}_x + I_y\dot{\omega}_y + I_z\dot{\omega}_z$. This we have just now evaluated by solving the determinant and $\overrightarrow{N}$ can be written as $N_x\hat{\imath} + N_y\hat{\jmath} + N_z\hat{k}$. Ok, so the component form of equation (6) is $I_x\dot{\omega}_x\hat{\imath} + I_y\dot{\omega}_y\hat{\jmath} + I_z\dot{\omega}_z\hat{k} +$ all of these terms which are written here and onto the right hand side, it is $N_x\hat{\imath} + N_y\hat{\jmath} + N_z\hat{k}$. Now, equating coefficients of $\hat{\imath},\ \hat{\jmath}, and\ \hat{k}$ on both the sides; we get; $I_x\dot{\omega}_x + \omega_y\ \omega_z\left(I_z - I_y\right) = N_x$; $I_y\dot{\omega}_y +\ \omega_x\ \omega_z(I_x - I_z) = N_y$; $I_z\dot{\omega}_z +\ \omega_x\ \omega_y\left(I_y - I_x\right) = N_z$. Let the body set of axes be the principal axes 1, 2, 3. Therefore above three equations can be rewritten as;

$I_1\dot{\omega}_1 + \omega_2\ \omega_3(I_3 - I_2) = N_1\ \ we\ call\ this\ as\ equatin\ number\ (7)$;

$I_2\dot{\omega}_2 +\ \omega_1\ \omega_3(I_1 - I_3) = N_2\ \ equatin\ number\ (8)$; $I_3\dot{\omega}_3 +\ \omega_1\ \omega_2(I_2 - I_1) = N_3\ \ equatin\ number\ (9)$.

All these three equations (7), (8) and (9) are called as Euler's equations of motion of a rigid body. If no external torque acts on a body, *i.e.,* $\vec{N} = 0$*;* the right hand sides of all of these three equations (7), (8) and (9) are zero. Therefore; we get; $I_1\dot{\omega}_1 + \omega_2\ \omega_3(I_3 - I_2) = 0\ we\ call\ this\ as\ eqaution\ number\ (10)$

$I_2\dot{\omega}_2 +\ \omega_1\ \omega_3(I_1 - I_3) = 0\ eqaution\ number\ (11)$; $I_3\dot{\omega}_3 +\ \omega_1\ \omega_2(I_2 - I_1) = 0\ eqaution\ number\ (12)$.

Now, multiplying each of these equations (10), (11) and (12) respectively by ω_1, ω_2 and ω_3 and then add them together. So, we are multiplying equation (10) by ω_1. So, it will be $I_1\dot{\omega}_1\omega_1$ and this is $\omega_1\omega_2\omega_3(I_3 - I_2)$. Similarly, we are multiplying equation #11 by ω_2, so it will be $I_2\dot{\omega}_2\omega_2$ and $\omega 1\omega 2\omega 3\left(I_1 - I_3\right)$; we multiply equation (12) by ω_3. So, the first term is $I_3\omega_3\dot{\omega}_3$ + $\omega 1\omega 2\omega 3$ and this brcket.

Now if you add them all of them after multiplying with $\omega 1, \omega 2, \omega 3$ respectively, we note that $\omega 1\omega 2\omega 3$ is common here and inside the bracket it will be $(I_3 - I_2) +\ (I_1 - I_3) + (I_2 - I_1)$. So, this will all get cancelled to give you 0. So only these three terms when they are multiplied with $\omega 1\omega 2\omega 3$ respectively will exist and we can write them as; $I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 = 0$. This same equation can be written as a time derivative of this bracket, that is, $\frac{d}{dt}\left[\frac{1}{2}\left(I_1\omega_1{}^2 + I_2\omega_2{}^2 + I_3\omega_3{}^2\right)\right]$ and that is all equal to 0.

Because if you solve this, differentiation of this you are getting back this equation. Now the quantity in the bracket over here is the kinetic energy T which is $\frac{1}{2}\ I_1\omega_1{}^2 + \frac{1}{2}\ I_2\omega_2{}^2 + \frac{1}{2}\ I_3\omega_3{}^2$. So, we can write; $\frac{dT}{dt} = 0$. If

$\frac{dT}{dt}$; time derivative of capital T; kinetic energy is 0 means capital T is constant or the kinetic energy is conserved.

Next, we can multiply now equations (10), (11), (12) instead of $\omega 1 \omega 2 \omega 3$ as in the previous case; now we multiply them with $I_1\omega_1$, $I_2\omega_2$ and $I_3\omega_3$ respectively and then add them together. So, the first term, which is already $I_1 \dot{\omega}_1$; when I multiply $I_1\omega_1$ here, it will be ${I_1}^2 \omega_1 \dot{\omega}_1$, and this will be $I_1 \omega_1 \omega_2\ \omega_3 (I_3 - I_2) = 0$. Similarly, we can write the next two terms. So, adding all of these together we get, we can also note here again that when you add all of these terms together, they will all go to 0. So only the first terms of these three equations will remain and we can add them to write; ${I_1}^2 \omega_1 \dot{\omega}_1 + {I_2}^2 \omega_2 \dot{\omega}_2 + {I_3}^2 \omega_3 \dot{\omega}_3 = 0$, which can also be expressed at the time derivative as half times $\frac{d}{dt}\left[{I_1}^2 {\omega_1}^2 + {I_2}^2 {\omega_2}^2 + {I_3}^2 {\omega_3}^2\right] = 0$. The quantity in the bracket is $[\vec{L}.\vec{L}]$. If you substitute $\vec{L}$ in its component form and (taking) take dot product with itself, we'll get that ${I_1}^2 {\omega_1}^2 + {I_2}^2 {\omega_2}^2 + {I_3}^2 {\omega_3}^2$. Therefore, half times $\frac{1}{2}\frac{d}{dt}[\vec{L}.\vec{L}]$ equal to 0. Since half cannot be equal to 0, so we can take this $\frac{d}{dt}[\vec{L}.\vec{L}]$ equals to 0. If you operate this onto the $[\vec{L}.\vec{L}]$ so we can (write), write it as the chain rule of differentiation or we can write $\vec{L}.\vec{L}$ equal to $\vec{L}\frac{d\vec{L}}{dt} + \frac{d\vec{L}}{dt}\vec{L}$, which is 2 times $\vec{L}\frac{d\vec{L}}{dt} = 0$. So, with this, we note that either $\vec{L}$ can be 0 or $\frac{d\vec{L}}{dt}$ is 0. Say $\vec{L}$ can be zero, we can get or $\frac{d\vec{L}}{dt}$ is 0. If $\frac{d\vec{L}}{dt}$ is zero means $\vec{L}$ is constant. So, angular momentum, either it can be 0 or it can be constant in the torque free motion. So the angular momentum is conserved.

These are the references for this module.

Thank you.