Quadrant II – Notes

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Module Name: Expression for 'Laplacian' operator in Cartesian Spherical Polar & Cylindrical Coordinate System

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Gradient, Divergence, Curl and Laplacian operator in SPC system:

All these operators we had expressed in Cartesian coordinate system and now our task (Not a simple one!) is to express in the spherical polar coordinate system.

The gradient of scalar function T is -

$$\mathbf{\nabla}\mathrm{T} = \left(\frac{\partial T}{\partial x}\,\hat{\imath} + \frac{\partial T}{\partial x}\,\hat{\jmath} + \frac{\partial T}{\partial x}\,\hat{k} \right) \qquad (25)$$

To transform gradient to SPC system, we can employ chain rule-

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial T}{\partial \theta}\left(\frac{\partial \theta}{\partial x}\right) + \frac{\partial T}{\partial \phi}\left(\frac{\partial \phi}{\partial x}\right). \qquad (26)$$

From inverse transformation equation, the bracketed partial differential coefficient can be worked out. Similarly, work out $\frac{\partial T}{\partial y}$ and $\frac{\partial T}{\partial z}$. It is a quite cumbersome job. There is nice technique to bypass conventional method and derive the desired expressions. (Again Griffith, treat it as holy book!!)

The indirect efficient approach is being discussed elaborately here. To begin with use arbitrary (orthogonal) curvilinear coordinates (u, v, w), developing formulas for the gradient, divergence, curl, and Laplacian in any such system. You can then specialize them to Cartesian, spherical, or cylindrical coordinates, or any other system you might wish to use. The table T() gives quick reference to generalized coordinates and corresponding coordinates in three systems.

The three unit vectors in generalized coordinate system will be $\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}$ *and* $\hat{w}$. The line element in this system - $dl = f\ du\ \hat{u} + \ g\ dv\ \hat{v} + \ h\ dw\ \hat{w}$

Table (4):

Generalized coordinate system	Cartesian coordinate system	Spherical polar coordinate system	Cylindrical coordinate system	f	g	h
U	X	R	R	1	1	1
V	Y	Θ	P	1	r	r sinθ
W	Z	Φ	Z	1	s	1

If you move from point (u, v, w) to point (u + du, v + dv, w + dw), a scalar function

T (u, v, w) changes by an amount,

$$dT = \frac{\partial T}{\partial u}\ du + \frac{\partial T}{\partial v}\ dv + \frac{\partial T}{\partial w}\ dw \qquad (23)$$

this is a standard theorem on partial differentiation. We can write it as a dot product,

$$dT = \ \boldsymbol{\nabla} T. dl = (\boldsymbol{\nabla} T)_u\ f\ du + \ (\boldsymbol{\nabla} T)_v\ f\ dv + \ (\boldsymbol{\nabla} T)_w\ f\ dw \qquad (24)$$

So that

$$(\boldsymbol{\nabla} T)_u = \ \frac{1}{f}\frac{\partial T}{\partial u} \qquad (25)$$

$$(\boldsymbol{\nabla} T)_v = \frac{1}{g}\ \frac{\partial T}{\partial v} \qquad (26)$$

$$(\boldsymbol{\nabla} T)_w = \ \frac{1}{h}\ \frac{\partial T}{\partial w} \qquad (27)$$

The gradient of T in generalized coordinate system can be written as –

$$\boldsymbol{\nabla} T \ \equiv \ \frac{1}{f}\ \frac{\partial T}{\partial u}\ \hat{u} + \ \frac{1}{g}\ \frac{\partial T}{\partial v}\ \hat{v} + \ \frac{1}{h}\ \frac{\partial T}{\partial w}\ \hat{w} \qquad (28)$$

Now referring to the table T(4), we can replace values of u, v, w and f, g, h by corresponding coordinates / values respectively for the three system to obtain expression for gradient.

$$(\nabla T)_{cartesian} = \frac{\partial T}{\partial x}\,\hat{\imath} + \frac{\partial T}{\partial y}\,\hat{\jmath} + \frac{\partial T}{\partial z}\,\hat{k}$$

$$(\nabla T)_{spherical} = \frac{\partial T}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\,\hat{\theta} + \frac{1}{r\,\sin\theta}\,\frac{\partial T}{\partial \phi}\,\hat{\phi}$$

Similar exercise has to be carried out in two steps to derive expression for divergence and curl in SPC system. Step 1) Find out the expression in generalized curvilinear coordinate system and step 2) Replacement of generalized coordinates u, v, w and f, g, h.

Divergence in Spherical polar coordinate system:

Since divergence is operated on the vector function, we must have one in generalized coordinate system. Let this vector function be-

$$\boldsymbol{A} = A_u\hat{\boldsymbol{u}} + A_v\hat{\boldsymbol{v}} + A_w\hat{\boldsymbol{w}}$$

We will look for the integral $\oint \boldsymbol{A}.\boldsymbol{da}$ and then take help of Gauss's divergence theorem to seek for relevant expression.

The area **da** for any surface bounded by the volume dτ in GEC system (Refer figure) be given by

$$da = -(gh)dv\,dw\,\hat{\boldsymbol{u}}$$

$$d\boldsymbol{\tau} = dl_u\,dl_v\,dl_w = (fgh)\,du\,dv\,dw$$

$$\boldsymbol{A}.\boldsymbol{da} = -A_u\,(gh)dv\,dw\,\hat{\boldsymbol{u}}$$

Consider a surface which is opposite the above surface. Then $A.da$ for this surface will be $+A_u\,(gh)dv\,dw\,\hat{\boldsymbol{u}}$ **.** The only difference is that $+A_u\,(gh)$ is being evaluated for u + du.

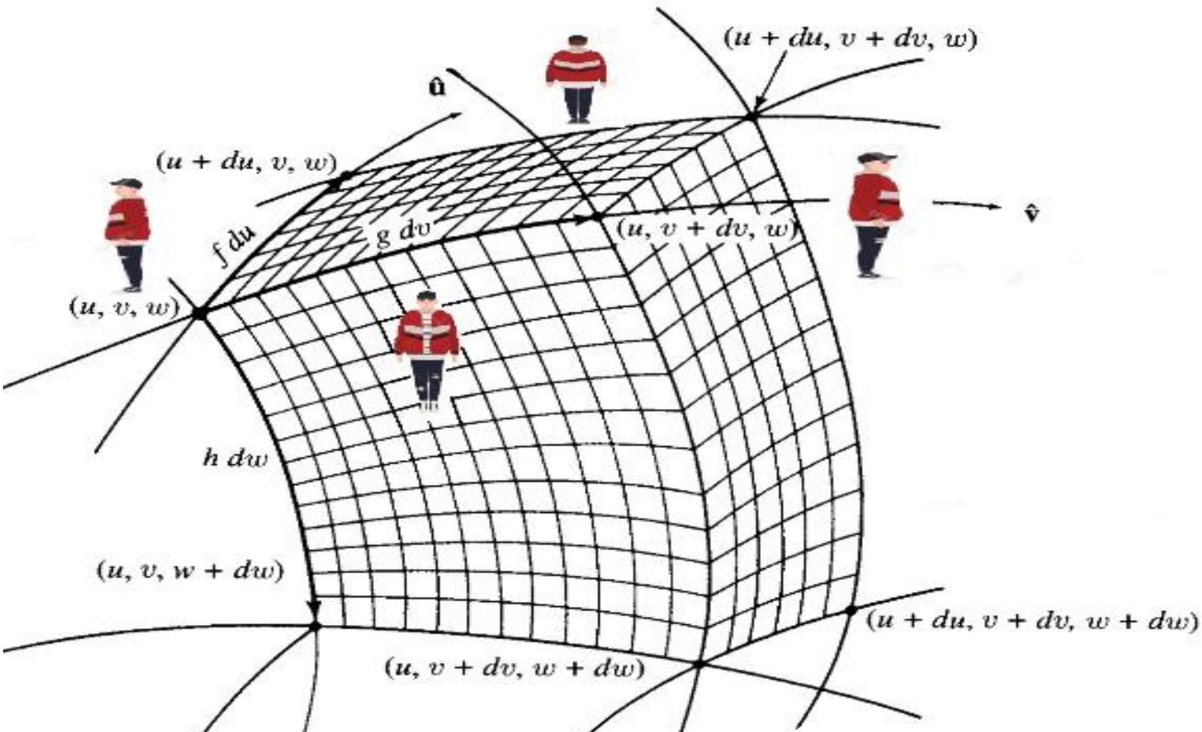
Since for any differentiable function dF -

$$F\,(u + du) - F\,(u) = \frac{\partial F}{\partial u}\,du, \tag{29}$$

The contribution from these two surfaces for the function $A_u\,gh$ spreading through area dv dw has to be :

$$\frac{\partial(\,ghA_u)}{\partial u}\,du\,dv\,dw = \frac{1}{fgh}\,\frac{\partial(ghA_u)}{\partial u}\,d\tau \tag{30}$$

Similarly, we consider other four surfaces, (imagine a cube that is enclosing a some kind of source and spreading vector function



through all the surface.) for which the amount of contribution will be –

$$\frac{1}{fgh}\,\frac{\partial(ghA_u)}{\partial v}\,d\tau \qquad (31)$$

$$\frac{1}{fgh}\frac{\partial(fgA_u)}{\partial w}\,d\tau$$

Then the total amount of contribution through all the surface will be –

$$\oint A.dl = \frac{1}{fgh}\left[\frac{\partial(ghA_u)}{\partial u} + \frac{\partial(fhA_u)}{\partial v} + \frac{\partial(fgA_u)}{\partial w}\right]d\tau \quad (32)$$

Applying Gauss's divergence theorem, L H S is equivalent to $\int \nabla.A\,d\tau$, thereby implying

$$\boldsymbol{\nabla}.\boldsymbol{A} = \frac{1}{fgh}\left[\frac{\partial(ghA_u)}{\partial u} + \frac{\partial(fhA_u)}{\partial v} + \frac{\partial(fgA_u)}{\partial w}\right] \quad (33)$$

Now we can express divergence in the spherical polar coordinate system with the help of table ( )

$$(\nabla.A)_{SPC\ system} = \frac{1}{r^2\sin\theta}\left[\frac{\partial\,(r^2\sin\theta\,A_r)}{\partial r} + \frac{\partial\,(r\sin\theta\,A_\theta)}{\partial\theta} + \frac{\partial(rA_\phi)}{\partial\phi}\right] \quad (34)$$

$$= \frac{1}{r^2}\frac{\partial\,(r^2\,A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\,(\sin\theta\,A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\,rA_\phi)}{\partial\phi} \quad (35)$$

Laplacian operator in SPC system :

Since the Laplacian of a scalar is by definition the divergence of the gradient, we can read off from Eqs. () and () the general formula for the ∇^2 operator.

$$\boldsymbol{\nabla}.\boldsymbol{\nabla}T = \frac{1}{fgh}\left[\frac{\partial(gh)}{\partial u}\,\hat{\boldsymbol{u}} + \frac{\partial(fh)}{\partial v}\,\hat{\boldsymbol{v}} + \frac{\partial(fg)}{\partial w}\,\hat{\boldsymbol{w}}\right] .\frac{1}{f}\frac{\partial T}{\partial u}\,\hat{\boldsymbol{u}} + \frac{1}{g}\frac{\partial T}{\partial v}\,\hat{\boldsymbol{v}} + \frac{1}{h}\frac{\partial T}{\partial w}\,\hat{w} \quad (36)$$

$$= \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial T}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{fh}{g}\frac{\partial T}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial t}{\partial w}\right)\right] \quad (37)$$

In spherical polar coordinate system –

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 t}{\partial \phi^2} \tag{38}$$