Hello students, welcome to module number 9 of unit 2 of section 2 of this paper. Mathematical Physics and Electromagnetic Theory one. The name of this module is Laplace's equation in one independent variable. I'm Yatin Desai Assistant Professor in Physics Chowgule College, Margao. In this module, you will learn about the first uniqueness theorem, the second uniqueness theorem and Laplace’s equation in one independent variable. At the end of this module you will be able to prove the first uniqueness theorem, you will be able to prove the second uniqueness theorem and also you'll be able to solve Laplace’s equation in one independent variable for rectangular cartesian coordinate system, spherical polar coordinate system and cylindrical polar coordinate system.

First we start with the first uniqueness theorem. The statement is if phi1, phi2, and so on, up to phi n are all solutions of Laplace's equation then phi equals C1phi1 plus C2phi2 plus so on up to Cn phi n is also a solution, where C1, C2, C3 and so on are the arbitrary constants, so phi1, phi2, phi3 are the solutions after multiplying those solutions with appropriate constants and then adding them together, we get the total solution.

To prove this, we take del square of phi. If we take del square of phi, we operate it on del square phi, which is given as this C1phi1 plus del square phi2 plus so one up to del square Cn phi n and since C1, C2 are all constants, we can bring it out of del square operator and write it as C1 del square phi1 + C2 del square phi2 plus so on up to Cn del square phi n.

Now this represents the Laplace's equation and we know that del square of the scalar potential in the Laplacian equation is 0. So all these terms on the right hand side are zero. So we get del square phi is also equal to 0, so that proves the statement that if phi1, phi2 and so on up to phi n are all the solutions of Laplace’s equation, then phi, which is the linear combination of C1phi1 plus C2phi2 and so on up to Cn phi n is also the solution.

Next, we discuss the second uniqueness theorem. Two solutions of Laplace’s equation that satisfy the same boundary condition differ at most by an additive constant. So to prove this we assume that let phi1and phi2 are two solutions of Laplace’s equation in some volume V not exterior to the surfaces S1, S2, S3 and so on up to Sn. phi1, phi2 satisfy the same boundary conditions on various surfaces and let phi equals phi1 minus phi2 such that, either phi or normal component of gradient of phi that is n cap dot del phi vanishes on the boundary.

So since phi is equal to phi1 minus phi2, it also follows that del square of phi which equals del square phi1 minus del square phi2 also equal to 0.

Now consider a vector which is phi del phi. Phi is the scalar but gradient of a scalar is a vector and to that I'm multiplying a scalar, so altogether this is a vector and we apply Gauss’s law to this vector. So which says that volume integral of divergence of a vector or a given volume is surface integral of normal component of the vector over a given surface. So integral over V not del dot phi del phi integrated over d tau. This is the divergence of this vector equals surface integral of phi del phi dot n cap integrated over d Sigma. So this is the normal component of the vector. Now we have chosen the solutions phi1 minus phi2 such that the normal component of this vector is zero. Therefore the volume integral del dot phi del phi d tau is equal to 0. We use one vector identity that del dot phi del phi is phi del square phi plus del phi whole square. And in the last equation, in the integrand, we had del dot phi del phi. But del square phi vanishes at all the points in V not it being the solution of the Laplace’s equation.

Therefore, we have volume integral of del phi whole square d tau equal to zero. Since del phi whole square must be either positive or zero at all points in V not and since it's integral is 0, only del phi whole square is 0 is only the possibility. If del phi whole square is 0. If you take a square root on both sides, then del phi equal to 0. That means a differentiation of phi is 0 means that five must be some constant and we write that phi as some constant C and phi we had written for phi1 minus phi2. Therefore phi 1 minus phi2 is equal to a constant C. Which proves second uniqueness theorem.

Next, we saw Laplace's equation in one independent variable. We do it first in the Cartesian coordinate system. In the Cartesian coordinate system, we write del square U equal to 0 as d square U by d x square equal to 0. Del square in one dimensional case,we write it as d square by dx square and it's a total differentiation. If you integrate it once, you'll get it as dU by dx equal to a (please read as a instead of C1). And if you integrate it again, you will get it as. a x + b, where a is a constant and b is also a constant. a and b are constants of integration, found either by specifying the values of the potential at two different points, or by specifying electric field at some point and the value of the potential at same point or some other point. So a and b, U = ax + b, represents the equation of a
straight line. That means U is increasing in space. In some cases, U may be either a constant or it may also decrease, but it cannot have a maxima and the minima at the intermediate points. It can have the maxima or the minima only at the endpoints, and this is also valid for the two dimensional case as well as the three dimensional case.

Next we obtain the solution of the Laplace’s equation in one dimension, by using spherical polar coordinates. To obtain the solution in one dimension r, we assume theta and phi are independent variables. So we write the equation as 1 by r square d by dr of r-square dU by dr equal to 0. If you multiply this r square onto the other side, it will go to zero. Therefore, d by d r of r square dU by dr, will be equal to 0 and we write r square d U by d r as a constant a.

If you integrate it r square dU by d r will be equal to a.So if you take that r square onto the other side, we will get it at du by d r as a r raised to minus 2. So, we further rearrange this equation and integrate it on both sides. So the integral of dU will be integral a r raised minus 2 d r. So it will be U = minus a r raised to minus 1 + b or U = minus a by r + b. where a and b are the constants.

We also obtain the solution of the Laplace’s equation in cylindrical polar coordinates. To obtain the solution in one dimension r, here we assume theta and z as independent variables. So we write that equation as 1 by r d by d r times r dU by d r = 0. So therefore it follows that d by d r of r dU by d r equal to 0 or integrating on both the sides we get r d U by d r equal to a.

Therefore, d U by d r equals a by r. And after integrating we get U as a integral of d r by r which is written as a ln of r + b. where a and b are the constants.

So we have obtained the solutions of Laplace’s equation in one dimension in case of the rectangular cartesian coordinate system, the spherical polar coordinate system and the cylindrical polar coordinate system.

These are the references for this module. Thank you.