

Quadrant II – Notes

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Module Name: Electric field inside a dielectric, Gauss's law in a dielectric, the electric Displacement vector

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Name of the Presenter: Dr. Swati R.Pawar, Associate Professor.

DCT's Dhempe College of Art's & Science, Miramar, Goa.

Notes:

The Electric Field inside a Dielectric

Before one can write an expression for the electric field inside a polarized medium, it is necessary to define this electric field precisely. What we are interested in, of course, is the macroscopic electric field, i.e., the average electric field in a small region of the dielectric which, nevertheless, contains a large number of molecules. An alternative and perhaps preferable approach is to define the electric field directly in terms of a macroscopic experiment: *the (macroscopic) electric field is the force per unit charge on a test charge embedded in the dielectric, in the limit where the test charge is so small that it does not itself affect the charge distribution.* This test charge must be dimensionally small from the macroscopic point of view (what we shall call a "point" charge), but it will be large compared with the size of a molecule

Although the above statement is the fundamental definition of the macroscopic electric field E , it is difficult to use this definition directly to obtain an expression for the field, since we would have to calculate the force on a charged body of

extended size, and then go to the limit as the size of the object decreased. Hence we find it expedient to use another property of the electric field to help us obtain the analytic expression we are seeking, and in this way we shall get \mathbf{E} in terms of the polarization charges of the medium.

The electrostatic field in a dielectric must have the same basic properties that we found scalar potential. Thus

$$\vec{\nabla} \times \vec{E} = \mathbf{0}$$

or equivalently,

$$\oint \vec{E} \cdot d\vec{l} = 0$$

Let us apply the last equation to the path ABCD shown in Fig. 1, where the segment AB lies in a needle-shaped cavity cut out of the dielectric, and the segment CD lies in the dielectric proper. Since the segments AD and BC may be made arbitrarily small, the line integral reduces to

$$\mathbf{E}_v \cdot \mathbf{I} - \mathbf{E}_d \cdot \mathbf{I} = \mathbf{0}$$

or equivalently.

$$\mathbf{E}_{vt} = \mathbf{E}_{dt}. \quad - \text{Eq(1)}$$

where the subscripts v and d refer to vacuum and dielectric respectively, and the subscript t stands for tangential component.

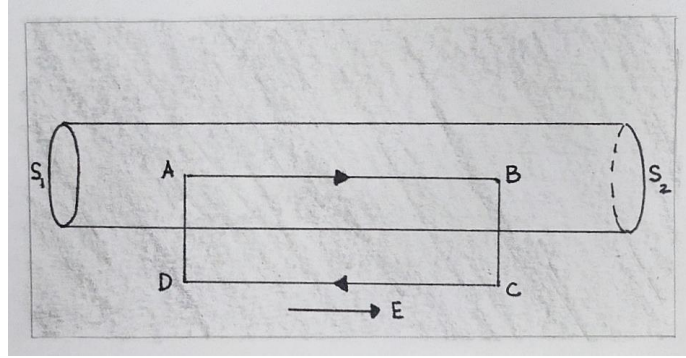


Fig.1

Eq(1) is valid regardless of the orientation of the needle-shaped cavity. If the "needle" is oriented along the direction of \mathbf{E} , then $\mathbf{E}_{vt} = \mathbf{E}_{dt}$; furthermore, by symmetry, the field in the cavity is along the direction of the needle,

that is, $\mathbf{E}_{vt} = \mathbf{E}_v$. We are thus led to an important conclusion:

The above statement is strictly true only for isotropic dielectrics. For anisotropic dielectrics the symmetry argument fails, and our conclusion must be generalized: the electric field in a dielectric is equal to the longitudinal component of the electric field inside a needle-shaped cavity in the dielectric provided the cavity axis is oriented parallel to the direction of the electric field in the dielectric.

The electric field in a dielectric is equal to the electric field inside a needle shaped cavity in the dielectric provided the cavity axis is oriented parallel to the direction of the electric field.

Evidently, the problem of calculating the electric field inside a dielectric reduces to calculating the electric field inside a needle-shaped cavity in the dielectric. But the electric field in the cavity is an external field, and hence may be determined by means of the results previously discussed. We assume here that the polarization of the dielectric is a given function $\mathbf{P}(x', y', z')$, and we calculate the potential and electric field arising from this polarization. Taking the field point \mathbf{r} at the center of the cavity and

using

$$\begin{aligned}\varphi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \left[\int_{S_0} \frac{\sigma_P d\mathbf{a}'}{|\mathbf{r} - \mathbf{r}'|} + \int_{V_0} \frac{\rho_P d\mathbf{v}'}{|\mathbf{r} - \mathbf{r}'|} \right], \\ &= \frac{1}{4\pi\epsilon_0} \int \frac{dq'_P}{|\mathbf{r} - \mathbf{r}'|}, \quad - \text{Eq(A)}\end{aligned}$$

we obtain for the potential

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{V_0 - V_1} \frac{\rho_P(x', y', z') d\mathbf{v}'}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{4\pi\epsilon_0} \int_{S_0 + S'} \frac{\sigma_P(x', y', z') d\mathbf{a}'}{|\mathbf{r} - \mathbf{r}'|}, -\text{Eq(2)}$$

where $V_0 - V_1$ is the volume of the dielectric excluding the "needle," S_0 is the exterior surface of the dielectric, and $S' = S_1 + S_2 + S_c$ are the needle surfaces. But from Fig. 3 it is seen that $\sigma_P = \mathbf{0}$ on the cylindrical surface S_c , of the needle; furthermore, the needle may be made arbitrarily thin so that the surfaces S_1 and S_2 have negligible area. Thus only the exterior surfaces of the dielectric contribute, and the surface integral of Eq(2) becomes identical in form to the surface integral of Eq(A). The volume integral Eq(2) excludes the cavity; however, the contribution of the cavity to this integral is negligible, as may readily be seen. The charge density ρ_P , is bounded; the quantity $d\mathbf{v}'/|\mathbf{r} - \mathbf{r}'|$ does not diverge at the field point (i.e., when $\mathbf{r}' = \mathbf{r}$) because the volume of a point is a higher-order zero than the $\lim|\mathbf{r} - \mathbf{r}'|$; and finally volume V_1 , of the needle may be made arbitrarily small by making the cavity thin. Thus we need not exclude the volume V_1 , and Eq(2) becomes similar in form to Eq(A). In other words, Eq(A) gives the potential $\varphi(\mathbf{r})$ regardless of whether the point \mathbf{r} is located inside or outside the dielectric.

Gauss's Law in a Dielectric.

The Electric Displacement

Gauss' law states that the electric flux across an arbitrary closed surface is proportional to the total charge enclosed by the surface. In applying Gauss' law to a region containing charges embedded in a dielectric, we must be careful to include all of the charge inside the gaussian surface, polarization charge as well as the free charges which are embedded.

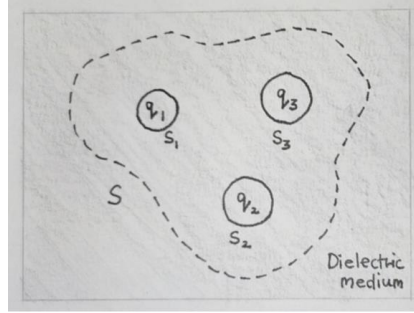


Fig.2

Above figure shows the dashed surface S is an imaginary closed surface located inside a dielectric medium. We embed a certain amount of charge, Q in the volume bounded by S , and it is assumed that charge exists on the surfaces of three conductors in amounts q_1 , q_2 and q_3 . By Gauss's law,

$$\oint_S \vec{E} \cdot \hat{n} da = \frac{1}{\epsilon_0} (Q + Q_P), \quad - \text{Eq(3)}$$

where Q is the net embedded charge, i.e.,

$$Q = q_1 + q_2 + q_3,$$

and Q_P , is the net polarization charge:

$$Q_P = \int_{S_1+S_2+S_3} \vec{P} \cdot \hat{n} da + \int_V (-\vec{\nabla} \cdot \vec{P}) dv. \quad - \text{Eq(4)}$$

Here V volume of the dielectric enclosed by S . There is no boundary of the dielectric material at S , so that the surface integral in Eq(4) *does not* contain a contribution from S .

If we transform the volume integral in Eq(4) to a surface integral by means of the divergence theorem, we must be careful to include contributions from all surfaces bounding V , namely, S , S_1 , S_2 and S_3 . It is evident that the last three contributions will cancel the first term of Eq(4), so that

$$Q_P = - \oint_S \mathbf{P} \cdot \mathbf{n} \, da. \quad - \text{Eq(5)}$$

Combining this result with Eq(3), we obtain

$$\oint_S (\epsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}) \cdot \hat{\mathbf{n}} \, da = Q. \quad - \text{Eq(6)}$$

Eq(6) states that the flux of the vector $\epsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}$ through a closed surface is equal to the net charge that we embedded in the volume enclosed by the surface. This vector quantity is important enough to give a name and a separate symbol. We define, therefore, a new macroscopic field $\vec{\mathbf{D}}$, the *electric displacement*:

$$\vec{\mathbf{D}} = \epsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}, \quad - \text{Eq(7)}$$

which evidently has the same units as $\vec{\mathbf{P}}$, charge per unit area.

In terms of $\vec{\mathbf{D}}$, Eq(6) becomes

$$\oint_S \vec{\mathbf{D}} \cdot \hat{\mathbf{n}} \, da = Q, \quad - \text{Eq(8)}$$

and this result is usually referred to as Gauss's law for the electric displacement, or simply Gauss's law. Eq(8) is applicable to a region of space bounded by any closed surface S ; if we apply it to a small region in which the charge enclosed is distributed as a charge density ρ , then Gauss's law becomes

$$\oint_S \vec{\mathbf{D}} \cdot \hat{\mathbf{n}} \, da = \rho \Delta V.$$

Dividing this equation by ΔV and proceeding to the limit, we obtain

$$\vec{\nabla} \cdot \vec{\mathbf{D}} = \rho, \quad - \text{Eq(9)}$$

a result which is sometimes called the differential form of Gauss's law.

The advantage of expressing the integral and differential forms of Gauss's law, Eq(8) and Eq(9), in terms of the vector \mathbf{D} is that only the charge Q or the charge density ρ that we embedded in the dielectric medium appears explicitly. This is what henceforth we shall usually call simply the charge (or the charge density). When it is necessary to distinguish it from the *polarization* charge Q_P the medium or the *total* charge $Q + Q_P$, the charge Q will be called the *external charge*. By "external" we do not mean that it necessarily lies outside the physical boundary of the piece of material; we mean that it is in addition to the charges that make up the atomic constitution of the neutral material.* Since in many problems the external charges are given, it is an advantage that the total electro static field at each point in the dielectric medium is expressed as the sum of two parts,

$$\vec{E}(x, y, z) = \frac{1}{\epsilon_0} \vec{D}(x, y, z) - \frac{1}{\epsilon_0} \vec{P}(x, y, z), \quad - \text{Eq(10)}$$

where the first term, $(1/\epsilon_0)\mathbf{D}$, is related to the external charge density through its divergence, and the second term, $(-1/\epsilon_0)\mathbf{P}$, is proportional to the polarization of the medium. In a vacuum the electric field is given entirely by the first term in Eq(10).
