Quadrant II – Transcript and Related Materials

Programme: Bachelor of Science (First Year)

Subject: Mathematics

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Unit: Unit 2: Real Sequences

Module Name: Properties of Subsequence

Module No: 62

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Glossary of terms/words:

Additional Examples/Illustrations

Properties of Subsequence

Property 1.If a sequence is convergent to limit l then all its subsequences also converge to same limit l.

For example : Consider sequence (x_n) where $x_n = (-1)^n$

Above sequence is not convergent because it's susequences (x_{2n}) and (x_{2n-1}) converges to two different limits 1 and -1 respectively

Property2. If every subsequence of (x_n) converge to l then the sequence (x_n) Itself converge to l

For Example : Consider (x_n) where $x_n = \frac{1}{n}$

All subsequences of (x_n) converges to 0

$\therefore$ (x_n) converges to 0

Theorem: Let (x_n) be a sequence in R.

If $\lim (x_{2n}) = l = \lim (x_{2n-1})$ then $\lim (x_n) = l$

Proof: Let $\epsilon > 0$ be any given number.

Since $x_{2n} \to l$,by definition , there exists k_1 ∈N such that whenever $\geq k_1$, $|x_{2k} - l| < \epsilon$ (i)

Since $x_{2n-1} \to l$,there exists k_2 ∈N such that

whenever $k \geq k_2$, $|x_{2k-1} - l| < \epsilon$ (ii)

Let k_0 =Max $\{k_1, k_2\}$ and let n_0 = $2k_0$

If $n \geq n_0$, then $n \geq 2k_0 \geq 2k_1$ and $n \geq 2k_0 \geq 2k_2$

Suppose n is even, n=2m say, then $2m \geq 2k_1$

Hence $m \geq k_1$

∴ by (i) , $|x_{2m} - l| < \epsilon$

Since $x_{2n-1} \to l$,there exists k_2 ∈N such that

whenever$k \geq k_2$, $|x_{2k-1} - l| < \epsilon$ (ii)

Let k_0 =Max $\{k_1, k_2\}$ and let n_0 = $2k_0$

If $n \geq n_0$, then $n \geq 2k_0 \geq 2k_1$ and $n \geq 2k_0 \geq 2k_2$

Suppose n is even, n=2m say, then $2m \geq 2k_1$

Hence $m \geq k_1$

∴ by (i) , $|x_{2m} - l| < \epsilon$

Thus from (iii) and (iv), we conclude that

if $n \geq n_0$, then $|x_n - l| < \epsilon$

Hence $x_n \to l$

i.e. lim (x_n)= l

Property 3: If (x_n) is any sequence of real numbers then it has a monotone subsequence

For Example: 1 Consider sequence (x_n) where x_n= n^2

i.e. (x_n) = (1,4,9,16,25,36,49,...........)

(x_{2n}) = (4,16,36,.....) is a monotone increasing subsequence of (x_n)

2. Consider sequence (x_n) where x_n = $\frac{1}{n}$ = (1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$,...........)

(x_{2n}) = ($\frac{1}{n}$, $\frac{1}{n}$, $\frac{1}{n}$,.......)is a monotone decreasing subsequence of (x_n)

Definition: Peak term of a sequence

Let (x_n) be a sequence in R.

Then m^{th} term x_m is said to be **a peak** if $x_m \geq x_n$ for all $n \geq m$

For Example: Consider (x_n) =(1, 0 ,$\frac{1}{2}$,2,3, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$,...............)

Consider x_1=1 By definition, it is **not a peak** since $x_1 \ngeqq x_n$ for all $n \geq m$

Similarly x_2=0 , x_3= $\frac{1}{2}$, x_4= 2 are **not peaks**

But x_5 =3 is **a peak**

Similarly $x_6 = \frac{1}{4}$, x_7 = $\frac{1}{5}$, … are **all peaks**

Remark : Any sequence can have **finite peaks** or **infinite peaks**

Monotone Subsequence Theorem

Theorem: Every real sequence has a monotone subsequence

Proof: Let (x_n) be any sequence of real numbers.

Then (x_n) can have finite peaks or infinite peaks

Case (i) Let (x_n) have infinitely many peaks say x_{m_1} , x_{m_2} x_{m_3} ……x_{m_k}…..

These subscripts are increasing order

i.e. $m_1 < \quad m_2 < \quad m_3 <$ ……… m_k …….

Clearly (x_{m_k}) is a monotone decreasing subsequence of (x_n)

Case (ii) Let (x_n) have finitely many peaks say x_{m_1} , x_{m_2} x_{m_3} ……x_{m_r}

and $m_1 < \quad m_2 < \quad m_3 <$ ……… m_r

(Assume next term to m_r is m_{r+1})

Let S_1 = m_{r+1}

∴ x_{S_1} is not a peak

i.e.we will find a term which is greater than x_{S_1}

Therefore, there exists $S_2 > S_1$ such that $x_{S_2} > x_{S_1}$

Again since x_{S_2} is not a peak, there exists $S_3 > S_2$ such that $x_{S_3} > x_{S_2}$

Hence we get $x_{S_3} > x_{S_2} > x_{S_1}$

Similarly we getx_{S_4}, x_{S_5} , x_{S_6},…….

Continuing this way we obtain a increasing subsequence (x_{S_n}) of (x_n)

i.e. (x_{S_n}) is a monotone increasing subsequence of (x_n)

Hence in both the cases (x_n) has monotone subsequence.