Programme : Bachelor of Science

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Paper Code : MTC 103

Paper Title : Differential Equations and Discrete Mathematics

Title of the Unit : Second Order Linear differential equations

Module Name : Problem when $Q(x) = e^{ax}$

Module No. : 38

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Outline

Learning Outcomes

After studying this module you should be able to

1. Identify the types of non-homogeneous terms for which the method of undetermined coefficients can be successfully applied.
2. Write the form of trial solutions when non-homogeneous terms is an exponential function.
3. Solve the second order linear differential equations with exponential function as non-homogeneous terms.

Second order Linear Non-homogeneous DE

$$L[y] = ay'' + by' + cy = Q(x) \tag{1}$$

where $a \neq 0$ b c are constants.

Undetermined Coefficients(UDC) -Constant coefficients

$Q(x)$ is of following types:

1. exponential functions
2. polynomial
3. sine or cosine functions
4. a combination of terms of types 1,2 and 3 above.

Method of UDC ($Q(x) = e^{ax}$)

Steps to solve the particular solution of (1) using UDC method when $Q(x)$ is an exponential function in $'x'$:

1. Find the complementary function $y_c(x)$ of homogeneous equation $L[y] = 0$.
2. Guess the trial solution $y_p(x)$ of (1) by listing all the terms of $Q(x)$ and its derivatives while ignoring the coefficients.
   - When $Q(x) = e^{ax}$; $y_p(x) = Ae^{ax}$.

Method of UDC [Cont...]

3. If a term of Q, say v, is also a term of the $y_c(x)$ corresponding to a root of $L[y] = 0$ occurring r times, then we multiply by x^r the trial solution $y_p(x)$ obtain from step(2) .
4. Compute $y_p''(x)$ $y_p'(x)$ and Substitue $y_p''(x),\ y_p'(x),\ y_p(x)$ in (1).
5. Determine constants by comparing like terms on both sides of the given equation (1).
6. Plug the values of the constants into y_p.

Hence, $y = y_c + y_p$ is the required general solution of (1).

Note

If $Q(x)$ is linear combination of two or more exponential functions, then due to linearity of (1), $y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \ldots y_{p_k}(x)$ *(Principle of Superposition)*

$Q(x)$ is an exponential function

Example (1)

Find the general solution of the differential equation:

$$y'' + 3y' + 2y = 3e^{x} \tag{2}$$

Solution:

The characteristic equation for the given differential equation is

$m^2 + 3m + 2 = 0$

$\implies m = -1, -2$

Thus, the complementary function $y_c(x)$ is $y_c(x) = c_1 e^{-x} + c_2 e^{-2x}$.

Example (Continued...)

$y_p(x) = Ae^x$

$y_p'(x) = Ae^x$

$y_p''(x) = Ae^x.$

Substituting the value of $y_p(x), y_p'(x), y_p''(x)$ in (2), we get

$$6Ae^x = 3e^x$$

Example (Continued...)

comparing the coefficients and solving we get,

$$A = \frac{1}{2}.$$

$\therefore\ y_p(x) = \frac{1}{2}e^x$

the primitive is

$y(x) = y_c(x) + y_p(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{2}e^x.$

$Q(x)$ is an exponential function

Example (2)

Find the general solution of the differential equation:

$$y'' - 3y' + 2y = e^x \qquad (3)$$

Solution:

The characteristic equation is $m^2 - 3m + 2 = 0$

$\implies m = 2, 1$

Thus, the complementary function $y_c(x)$ is $y_c(x) = c_1 e^x + c_2 e^{2x}$.

Example (Continued...)

$y_p(x) = x(Ae^x)$

$y_p'(x) = xAe^x + Ae^x$

$y_p''(x) = xAe^x + 2Ae^x.$

Substituting the value of $y_p(x), y_p'(x), y_p''(x)$ in (3), we get
$(xAe^x + 2Ae^x) - 3(xAe^x + Ae^x) + 2x(Ae^x) = e^x$
$-Ae^x = e^x$

Example (Continued...)

Comparing the coefficients we get,

$$A = -1.$$

$\therefore y_p(x) = -xe^x$

the primitive is $y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 e^{2x} - xe^x$.

$Q(x)$ is an exponential function

Example (3)

Find the general solution of the differential equation:

$$y'' - 4y' + 4y = e^{2x} \tag{4}$$

Solution:

The characteristic equation is $m^2 - 4m + 4 = 0 \Longrightarrow \; m = 2, 2$

Thus, the complementary function $y_c(x)$ is $y_c(x) = (c_1 + c_2 x)e^{2x}$.

Example (Continued...)

$y_p(x) = x^2(Ae^{2x})$

$y_p'(x) = 2x^2Ae^{2x} + 2xAe^{2x}$

$y_p''(x) = 4x^2Ae^{2x} + 8xAe^{2x} + 2Ae^{2x}.$

Substituting the value of $y_p(x), y_p'(x), y_p''(x)$ in (4), we get

$(4x^2Ae^{2x} + 8xAe^{2x} + 2Ae^{2x}) - 4(2x^2Ae^{2x} + 2xAe^{2x}) + 4x^2(Ae^{2x}) = e^{2x}$

Example (Continued...)

comparing the coefficients we get, $A = \frac{1}{2}$.

$\therefore\ y_p(x) = \frac{1}{2}x^2e^{2x}$,

the primitive is $y(x) = y_c(x) + y_p(x) = (c_1 + c_2x)e^{2x} + \frac{1}{2}x^2e^{2x}$.

$Q(x)$ is an exponential function

Example (4)

Find the general solution of the differential equation:

$$y'' + y' - 2y = 4e^{x} + 3e^{-2x} \tag{5}$$

Solution:

The characteristic equation is $m^2 + m - 2 = 0$

$\implies m = -2, 1$

Thus, the complementary function $y_c(x)$ is $y_c(x) = c_1 e^{-2x} + c_2 e^{x}$

Example (Continued...)

$y_p(x) = x(Ae^x + Be^{-2x})$

$y_p'(x) = xAe^x + Ae^x - 2xBe^{-2x} + Be^{-2x}$

$y_p''(x) = xAe^x + 2Ae^x + 4xBe^{-2x} - 4Be^{-2x}$.

Substituting the value of $y_p(x), y_p'(x), y_p''(x)$ in (5), we get
$3Ae^x - 3Be^{-2x} = 4e^x + 3e^{-2x}$

Example (Continued...)

Comparing the coefficients we get, $A = \frac{4}{3}, \quad B = -1.$

$\therefore y_p(x) = \frac{4}{3}xe^x - xe^{-2x}$

the primitive is

$y(x) = y_c(x) + y_p(x) = c_1 e^{-2x} + c_2 e^x + \frac{4}{3}xe^x - xe^{-2x}.$

$Q(x)$ is an exponential function

Example (5)

Find the general solution of the differential equation:

$$y'' - 2y' + y = e^x + 4 \tag{6}$$

Solution:

The characteristic equation is $m^2 - 2m + 1 = 0 \implies m = 1, 1$

Thus, the complementary function $y_c(x)$ is $y_c(x) = (c_1 + xc_2)e^x$.

Example (Continued...)

$y_p(x) = x^2Ae^x + B$

$y_p'(x) = x^2Ae^x + 2xAe^x$

$y_p''(x) = x^2Ae^x + 4xAe^x + 2Ae^x$

Substituting the value of $y_p(x), y_p'(x), y_p''(x)$ in (6),we get

$$2Ae^x + B = e^x + 4$$

Example (Continued...)

Comparing the coefficients we get, $A = \frac{1}{2}, \;\; B = 4.$

$\therefore y_p(x) = (\frac{x^2}{2})e^x + 4,$

the primitive is $y(x) = y_c(x) + y_p(x) = (c_1 + xc_2)e^x + (\frac{x^2}{2})e^x + 4.$

References I

Martin Braun and Martin Golubitsky.
Differential equations and their applications, volume 1.
Springer, 1983.

Richard Bronson.
Schaum's outline of theory and problems of differential equations.
McGraw-Hill, 1994.

EA Coddington.
An introduction to ordinary differential equations, phi learning pvt.
Ltd., New Delhi, 2012.

BS Grewal.
Higher engg. mathematics, 2006.

References II

George F Simmons.
Differential equations with applications and historical notes.
CRC Press, 2016.

Thank You