Programme : Bachelor of Science

Subject : Mathematics

Semester : III

Paper Code : MTC 103

Paper Title : Differential Equations and Discrete Mathematics

Title of the Unit : Second Order Linear differential equations

Module Name : Problem when Q(x) = sinax (or cosax)

Module No. : 40

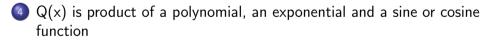
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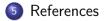
Outline

Introduction

2 Method of UDC (Q(x) is sine or cosine function)

3 Examples





Learning Outcomes

After studying this module you should be able to

- Identify the types of non-homogeneous terms for which the method of undetermined coefficients can be successfully applied.
- Write the form of trial solutions when non-homogeneous terms are sine or cosine functions.
- Solve the second order linear differential equations with sine or cosine functions as non-homogeneous terms.

Second order Linear Non-homogeneous DE

$$L[y] = ay'' + by' + cy = Q(x)$$
 (1)

where $a \neq 0$ b c are constants. Undetermined Coefficients(UDC) -Constant coefficients Q(x) is of following types:

- exponential functions
- 2 polynomial
- **③** sine or cosine functions
- a combination of terms of types 1,2 and 3 above.

Method of UDC

Steps to solve the particular solution of (1) using UDC method when Q(x) is a sine or cosine function:

- 1 Find the complementary function $y_c(x)$ of homogeneous equation L[y] = 0.
- 2 Guess the trial solution $y_p(x)$ of (1)by list all the terms of Q(x) and its derivatives while ignoring the coefficients.
 - When $Q(x) = \sin ax(or \cos ax)$; $y_p(x) = A \sin ax + B \cos ax$.

Method of UDC [Continued...]

- 3 If a term of Q, say v, is also a term of the $y_c(x)$ corresponding to a root of L[y] = 0 occurring r times, then we multiply by x^r the trial solution $y_p(x)$ obtain from step(2).
- 4 Compute $y_p''(x) y_p'(x)$ and Substitue $y_p''(x), y_p'(x), y_p(x)$ in (1).
- 5 Determine constants by comparing like terms on both sides of the given equation (1).
- 6 Plug the values of the constants into y_p .

Hence, $y = y_c + y_p$ is the required general solution of (1).

Note

If Q(x) is linear combination of two or more functions, then due to linearity of (1), $y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \dots + y_{p_k}(x)$ (Principle of Superposition)

Example (1)

Find the general solution of the differential equation:

$$y'' + 4y = \sin x$$

(2)

Solution:

The characteristic equation for the given differential equation is $m^2 + 4 = 0 \implies m = 2i, -2i$ Thus, the complementary function $y_c(x)$ can be expressed as $y_c(x) = c_1 \cos 2x + c_2 \sin 2x$

 $y_p(x) = A\sin x + B\cos x$

$$y'_p(x) = A\cos x - B\sin x$$

$$y_p''(x) = -A\sin x - B\cos x.$$

Substituting the value of $y_p(x), y'_p(x), y''_p(x)$ in (2)

 $3A\sin x + 3B\cos x = \sin x$

Comparing the coefficients and solving we get, $A = \frac{1}{3}$, B = 0.

$$\therefore y_p(x) = \frac{1}{3} \sin x,$$

the primitive is $y(x) = y_c(x) + y_p(x) = c_1 \sin 2x + c_2 \cos 2x + \frac{1}{3} \sin x$.

Example (2)

Find the general solution of the differential equation:

$$y'' + y = \sin x - \cos x$$

(3)

Solution:

The characteristic equation for the given differential equation is $m^2 + 1 = 0 \implies m = i, -i$ Thus, the complementary function $y_c(x)$ can be expressed as $y_c(x) = (c_1 \cos x + c_2 \sin x)$

 $y_p(x) = x(A\sin x + B\cos x)$

$$y'_p(x) = (A\sin x + B\cos x) + x(A\cos x - B\sin x)$$

$$y_p''(x) = 2(A\cos x - B\sin x) - x(A\sin x + B\cos x)$$

Substituting the value of $y_p(x), y'_p(x), y''_p(x)$ in (3),

 $2A\cos x - 2B\sin x = \sin x - \cos x.$

Comparing the coefficients we get, $A = -\frac{1}{2} B = -\frac{1}{2}$.

$$\therefore y_p(x) = -\frac{1}{2}x(\sin x + \cos x)$$

the primitive is $y(x) = y_c(x) + y_p(x) = (c_1 \cos x + c_2 \sin x) - \frac{1}{2}x(\sin x + \cos x).$

Example (3)

Find the general solution of the differential equation:

$$y'' - y' = 5e^x - \sin 2x$$

(4)

Solution:

The characteristic equation for the given differential equation is $m^2 - m = 0 \implies m = 0, 1$ Thus, the complementary function $y_c(x)$ can be expressed as $y_c(x) = c_1 + c_2 e^x$

$$y_p(x) = x(Ae^x) + B\sin 2x + C\cos 2x$$

$$y'_p(x) = xAe^x + Ae^x + 2B\cos 2x - 2C\sin 2x$$

$$y_{p}''(x) = xAe^{x} + 2Ae^{x} - 4B\sin 2x - 4C\cos 2x.$$

Substituting the value of $y_p(x), y'_p(x), y''_p(x)$ in (4),

Comparing the coefficients we get, A = 5, $B = \frac{1}{5} C = -\frac{1}{10}$.

$$\therefore y_p(x) = 5xe^x + \frac{1}{5}\sin 2x - \frac{1}{10}\cos 2x$$

the primitive is $y(x) = y_c(x) + y_p(x) = c_1 + c_2 e^x + 5x e^x + \frac{1}{5} \sin 2x - \frac{1}{10} \cos 2x.$ If Q(x) = P_k(x)e^{αx} sin βx or Q(x) = P_k(x)e^{αx} cos βx, then
If α + iβ is not a root of the characteristic equation corresponding to (1). The trial solution

$$y_{p}(x) = [A_{0}x^{k} + A_{1}x^{k-1} + \ldots + A_{k-1}x + A_{k}]e^{\alpha x} \cos \beta x + [B_{0}x^{k} + B_{1}x^{k-1} + \ldots + B_{k-1}x + B_{k}]e^{\alpha x} \sin \beta x$$

2 If $\alpha + i\beta$ is a root of the characteristic equation corresponding to (1) and say, r- times repeated root then the modified trial solution is

$$y_{\rho}(x) = x^{r} \{ [A_{0}x^{k} + A_{1}x^{k-1} + \ldots + A_{k-1}x + A_{k}] e^{\alpha x} \cos \beta x + [B_{0}x^{k} + B_{1}x^{k-1} + \ldots + B_{k-1}x + B_{k}] e^{\alpha x} \sin \beta x \}.$$

Example (4)

Find the general solution of the differential equation:

$$y''-2y'=e^x\sin x$$

(5)

Solution:

The characteristic equation for the given differential equation is $m^2 - 2m = 0 \implies m = 0, 2$ Thus, the complementary function $y_c(x)$ can be expressed as $y_c(x) = c_1 + c_2 e^{2x}$

Example (Contd...)

 $y_p(x) = Ae^x \sin x + Be^x \cos x$

 $y'_p(x) = Ae^x \sin x + Be^x \cos x + Ae^x \cos x + Be^x \sin x$

 $y_p''(x) = -2Be^x \sin x + 2Ae^x \cos x.$

Substituting the value of $y'_p(x), y''_p(x)$ in (5)

$$(-2A-2B)e^x \sin x + (2A-2B)e^x \cos x = e^x \sin x$$

Example (Contd...)

Comparing the coefficients we get, $A = -\frac{1}{4}, B = -\frac{1}{4}.$ $\therefore y_p(x) = -\frac{1}{4}e^x \sin x - \frac{1}{4}\cos x.$ the primitive is $y(x) = y_c(x) + y_p(x) = c_1 + c_2e^x - \frac{1}{4}e^x \sin x - \frac{1}{4}\cos x.$

Example (5)

Write the form of the trial solution:

$$1 y''(x) + 2y' + 3y = x \cos 3x - \sin 3x.$$

Solution: $y_p = (A + Bx) \cos 3x + (C + Dx) \sin 3x$.

Example

Write the form of the trial solution:

2
$$y'' + 2y' + 5y = xe^{-x}\cos 2x$$

Solution:
$$y_p = x[(A + Bx)e^{-x}\cos 2x + (C + Dx)e^{-x}\sin 2x]$$

Example

Write the form of the trial solution :

3
$$y'' - 5y' + 6y = xe^x \cos 2x$$
.

Solution: $y_p = e^x[(Ax + B)\cos 2x + (Cx + D)\sin 2x]$

Example

Write the form of the trial solution :

$$4 y'' + y = x^2 \sin x.$$

Solution: $y_p = x[(A + Bx + Cx^2)\cos x + (D + Ex + Fx^2)\sin x]$

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Thank You