Programme	:	Bachelor of Science
Subject	:	Mathematics
Semester	:	III
Paper Code	:	MTC 103
Paper Title	:	Differential Equations and Discrete Mathematics

Title of the Unit : Second Order Linear differential equations

Module Name : Problem when Q(x) =sinax (or cosax)

Module No. : 40

Dr. Mamta Kumari
Assistant Professor
Department of Mathematics
DCT's Dhempe College of Arts and Science
Miramar,Panaji-Goa.

Outline

Learning Outcomes

After studying this module you should be able to

1. Identify the types of non-homogeneous terms for which the method of undetermined coefficients can be successfully applied.
2. Write the form of trial solutions when non-homogeneous terms are sine or cosine functions.
3. Solve the second order linear differential equations with sine or cosine functions as non-homogeneous terms.

Second order Linear Non-homogeneous DE

$$L[y] = ay'' + by' + cy = Q(x) \tag{1}$$

where $a \neq 0$ b c are constants.

Undetermined Coefficients(UDC) -Constant coefficients

$Q(x)$ is of following types:

1. exponential functions
2. polynomial
3. sine or cosine functions
4. a combination of terms of types 1,2 and 3 above.

Method of UDC

Steps to solve the particular solution of (1) using UDC method when $Q(x)$ is a sine or cosine function:

1. Find the complementary function $y_c(x)$ of homogeneous equation $L[y] = 0$.
2. Guess the trial solution $y_p(x)$ of (1)by list all the terms of $Q(x)$ and its derivatives while ignoring the coefficients.
    - When $Q(x) = \sin ax (or \cos ax)$; $y_p(x) = A \sin ax + B \cos ax$.

Method of UDC [Continued...]

3 If a term of Q, say v, is also a term of the $y_c(x)$ corresponding to a root of $L[y] = 0$ occurring r times, then we multiply by x^r the trial solution $y_p(x)$ obtain from step(2) .

4 Compute $y_p''(x)$ $y_p'(x)$ and Substitue $y_p''(x),\ y_p'(x),\ y_p(x)$ in (1).

5 Determine constants by comparing like terms on both sides of the given equation (1).

6 Plug the values of the constants into y_p.

Hence, $y = y_c + y_p$ is the required general solution of (1).

Note

If $Q(x)$ is linear combination of two or more functions, then due to linearity of (1), $y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \ldots y_{p_k}(x)$ (Principle of Superposition)

Example (1)

Find the general solution of the differential equation:

$$y'' + 4y = \sin x \tag{2}$$

Solution:

The characteristic equation for the given differential equation is
$m^2 + 4 = 0 \Longrightarrow \; m = 2i, -2i$
Thus, the complementary function $y_c(x)$ can be expressed as
$y_c(x) = c_1 \cos 2x + c_2 \sin 2x$

Example (Continued...)

$y_p(x) = A\sin x + B\cos x$

$y_p'(x) = A\cos x - B\sin x$

$y_p''(x) = -A\sin x - B\cos x.$

Substituting the value of $y_p(x), y_p'(x), y_p''(x)$ in (2)

$$3A\sin x + 3B\cos x = \sin x$$

Example (Continued...)

Comparing the coefficients and solving we get, $A = \frac{1}{3}, \ B = 0.$

$\therefore \ y_p(x) = \frac{1}{3}\sin x,$

the primitive is $y(x) = y_c(x) + y_p(x) = c_1 \sin 2x + c_2 \cos 2x + \frac{1}{3}\sin x.$

Example (2)

Find the general solution of the differential equation:

$$y'' + y = \sin x - \cos x \tag{3}$$

Solution:

The characteristic equation for the given differential equation is
$m^2 + 1 = 0 \implies m = i, -i$
Thus, the complementary function $y_c(x)$ can be expressed as
$y_c(x) = (c_1 \cos x + c_2 \sin x)$

Example (Continued...)

$y_p(x) = x(A\sin x + B\cos x)$

$y_p'(x) = (A\sin x + B\cos x) + x(A\cos x - B\sin x)$

$y_p''(x) = 2(A\cos x - B\sin x) - x(A\sin x + B\cos x)$

Substituting the value of $y_p(x), y_p'(x), y_p''(x)$ in (3),

$$2A\cos x - 2B\sin x = \sin x - \cos x.$$

Example (Continued...)

Comparing the coefficients we get, $A = -\frac{1}{2}$ $B = -\frac{1}{2}$.

$\therefore\ y_p(x) = -\frac{1}{2}x(\sin x + \cos x)$

the primitive is
$y(x) = y_c(x) + y_p(x) = (c_1 \cos x + c_2 \sin x) - \frac{1}{2}x(\sin x + \cos x)$.

Example (3)

Find the general solution of the differential equation:

$$y'' - y' = 5e^{x} - \sin 2x \tag{4}$$

Solution:

The characteristic equation for the given differential equation is
$m^2 - m = 0 \implies m = 0, 1$
Thus, the complementary function $y_c(x)$ can be expressed as
$y_c(x) = c_1 + c_2 e^{x}$

Example (Continued...)

$$y_p(x) = x(Ae^x) + B\sin 2x + C\cos 2x$$

$$y_p'(x) = xAe^x + Ae^x + 2B\cos 2x - 2C\sin 2x$$

$$y_p''(x) = xAe^x + 2Ae^x - 4B\sin 2x - 4C\cos 2x.$$

Substituting the value of $y_p(x), y_p'(x), y_p''(x)$ in (4),

Example (Continued...)

Comparing the coefficients we get, $A = 5,\ B = \frac{1}{5}\ C = -\frac{1}{10}$.

$\therefore\ y_p(x) = 5xe^x + \frac{1}{5}\sin 2x - \frac{1}{10}\cos 2x$

the primitive is
$y(x) = y_c(x) + y_p(x) = c_1 + c_2e^x + 5xe^x + \frac{1}{5}\sin 2x - \frac{1}{10}\cos 2x.$

If $Q(x) = P_k(x)e^{\alpha x}\sin\beta x$ or $Q(x) = P_k(x)e^{\alpha x}\cos\beta x$, then

1. If $\alpha + i\beta$ is not a root of the characteristic equation corresponding to (1).
   The trial solution

$$\begin{aligned} y_p(x) &= [A_0x^k + A_1x^{k-1} + \ldots + A_{k-1}x + A_k]e^{\alpha x}\cos\beta x \\ &\quad +[B_0x^k + B_1x^{k-1} + \ldots + B_{k-1}x + B_k]e^{\alpha x}\sin\beta x \end{aligned}$$

2. If $\alpha + i\beta$ is a root of the characteristic equation corresponding to (1)and say, $r-$ times repeated root then the modified trial solution is

$$\begin{aligned} y_p(x) &= x^r\{[A_0x^k + A_1x^{k-1} + \ldots + A_{k-1}x + A_k]e^{\alpha x}\cos\beta x \\ &\quad +[B_0x^k + B_1x^{k-1} + \ldots + B_{k-1}x + B_k]e^{\alpha x}\sin\beta x\}. \end{aligned}$$

Example (4)

Find the general solution of the differential equation:

$$y'' - 2y' = e^x \sin x \tag{5}$$

Solution:

The characteristic equation for the given differential equation is

$m^2 - 2m = 0 \implies m = 0, 2$

Thus, the complementary function $y_c(x)$ can be expressed as

$y_c(x) = c_1 + c_2 e^{2x}$

Example (Contd...)

$y_p(x) = Ae^x \sin x + Be^x \cos x$

$y_p'(x) = Ae^x \sin x + Be^x \cos x + Ae^x \cos x + Be^x \sin x$

$y_p''(x) = -2Be^x \sin x + 2Ae^x \cos x.$

Substituting the value of $y_p'(x), y_p''(x)$ in (5)

$$(-2A - 2B)e^x \sin x + (2A - 2B)e^x \cos x = e^x \sin x.$$

Example (Contd...)

Comparing the coefficients we get,
$A = -\frac{1}{4},\ B = -\frac{1}{4}.$

$\therefore y_p(x) = -\frac{1}{4} e^x \sin x - \frac{1}{4} \cos x.$

the primitive is $y(x) = y_c(x) + y_p(x) = c_1 + c_2 e^x - \frac{1}{4} e^x \sin x - \frac{1}{4} \cos x.$

Example (5)

Write the form of the trial solution:

1. $y''(x) + 2y' + 3y = x\cos 3x - \sin 3x.$

Solution: $y_p = (A + Bx)\cos 3x + (C + Dx)\sin 3x.$

Example

Write the form of the trial solution:

2 $y'' + 2y' + 5y = xe^{-x}\cos 2x$

Solution: $y_p = x[(A + Bx)e^{-x}\cos 2x + (C + Dx)e^{-x}\sin 2x]$

Example

Write the form of the trial solution :

3 $y'' - 5y' + 6y = xe^x \cos 2x.$

Solution: $y_p = e^x[(Ax + B)\cos 2x + (Cx + D)\sin 2x]$

Example

Write the form of the trial solution :

4 $y'' + y = x^2 \sin x$.

Solution: $y_p = x[(A + Bx + Cx^2)\cos x + (D + Ex + Fx^2)\sin x]$

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Thank You