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Subject	:	Mathematics
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Paper Title	:	Differential Equations and Discrete Mathematics

Title of the Unit : Second Order Linear differential equations

Module Name : Linear ODEs with variable coefficients

Module No. : 14

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Outline

Learning Outcomes

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2. Write the Cauchy-Euler non-homogeneous DEs as linear DEs with constant coefficients

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1. Identify the Cauchy-Euler non-homogeneous DEs.
2. Write the Cauchy-Euler non-homogeneous DEs as linear DEs with constant coefficients
3. Solve the the Cauchy-Euler non-homogeneous DEs of second order using undetermined coefficients method.

Cauchy-Euler Equations

A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots a_0 y = g(x), \qquad (1)$$

where the coefficients $a_n, a_{n-1}, \dots, a_0$ are constants, is known as a Cauchy-Euler equation.

Note

The degree $k = n, n-1, \dots, 1, 0$ of the monomial coefficients x^k matches the order k of differentiation $\frac{d^k y}{dx^k}$.

Second order Non-homogeneous Cauchy-Euler DEs

Consider second order non-homogeneous Cauchy-Euler equation

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = Q(x), \tag{2}$$

where a, b, and c are constants.

Method

To reduce equation (2) to the linear homogeneous differential equation with constant coefficients. Substitute $x = e^t$ *or* $t = \log x$. Then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$\therefore$

$$x\frac{dy}{dx} = \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{1}{e^t}\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{1}{e^t}\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$\therefore$,

$$x^2\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Steps to find the solution

1. Reduce the equation (2) to linear DE with constant coefficients.
2. Find the general solution of the homogeneous Euler equation i.e y_c.
3. Using the method of undetermined coefficients or the method of variation of parameters, find a particular solution depending on the right side of the given non-homogeneous DE i.e. y_p.
4. The general solution of equation (2) is

$$y = y_c + y_p$$

Example (1)

Find the general solution of the differential equation:

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x \tag{3}$$

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$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x \tag{3}$$

Solution:

Let $x = e^t$ *or* $t = \log x$ then

$$x\frac{dy}{dx} = \frac{dy}{dt}; \quad x^2\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Example (Continued...)

Then the given equation (3) becomes

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t. \tag{4}$$

The characteristic equation is

$m^2 - 2m + 1 = 0$

Example (Continued...)

Then the given equation (3) becomes

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t. \tag{4}$$

The characteristic equation is

$m^2 - 2m + 1 = 0 \implies m = 1, 1$

Thus, the complementary function $y_c(x)$ can be expressed as
$y_c(t) = (c_1 + c_2 t)e^t$

Example (Continued...)

$y_p(t) = At + B$

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$y_p''(t) = 0$

Example (Continued...)

$y_p(t) = At + B$

$y_p'(t) = A$

$y_p''(t) = 0$

Substituting the value of $y_p(t), y_p'(t), y_p''(t)$ in (4)

$$-2A + At + B = t$$

Comparing the coefficients and solving we get, $A = 1,\ B = 2.$

Example (Continued...)

$\therefore y_p(t) = t + 2,$

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the primitive is $y(t) = y_c(t) + y_p(t) = (c_1 + c_2 t)e^t + t + 2.$

$\therefore y(x) = (c_1 + c_2 \log x)x + \log x + 2.$

Example (2)

Find the general solution of the differential equation:

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \frac{1}{x^2} + 2\log x. \tag{5}$$

Example (2)

Find the general solution of the differential equation:

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \frac{1}{x^2} + 2\log x. \qquad (5)$$

Solution:

Let $x = e^t$ *or* $t = \log x$ then

$$x\frac{dy}{dx} = \frac{dy}{dt}; \quad x^2\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Example (Continued...)

Then the given equation (5) becomes

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-2t} + 2t \qquad (6)$$

The characteristic equation is

$m^2 + 3m + 2 = 0$

Example (Continued...)

Then the given equation (5) becomes

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-2t} + 2t \tag{6}$$

The characteristic equation is

$m^2 + 3m + 2 = 0 \implies m = -2, -1$

Thus, the complementary function $y_c(x)$ can be expressed as
$y_c(t) = (c_1 e^{-t} + c_2 e^{-2t})$

Example (Continued...)

$y_p(t) = t(Ae^{-2t}) + Bt + C$

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Example (Continued...)

$y_p(t) = t(Ae^{-2t}) + Bt + C$

$y_p'(t) = -2t(Ae^{-2t}) + Ae^{-2t} + B$

$y_p''(t) = 4t(Ae^{-2t}) - 4(Ae^{-2t})$

Substituting the value of $y_p(t), y_p'(t), y_p''(t)$ in (6)

$$-Ae^{-2t} + 2Bt + 3B + 2c = e^{-2t} + 2t$$

Comparing the coefficients and solving we get, $A = -1,\ B = 1\ C = \frac{-3}{2}$.

Example (Continued...)

$\therefore y_p(t) = -te^{-2t} + t - \frac{3}{2},$

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$\therefore y_p(t) = -te^{-2t} + t - \frac{3}{2}$,

the primitive is $y(t) = y_c(t) + y_p(t) = (c_1 e^{-t} + c_2 e^{-2t}) - te^{-2t} + t - \frac{3}{2}$,

$\therefore y(x) = \left(\frac{c_1}{x} + \frac{c_2}{x^2}\right) - \frac{\log x}{x^2} + \log x - \frac{3}{2}$.

A Different Form

A second order equation of the form

$$a(x-x_0)^2\frac{d^2y}{dx^2}+b(x-x_0)\frac{dy}{dx}+cy=0 \tag{7}$$

is also a Cauchy-Euler equation.Observe that (7) reduces to (2) when $x_0 = 0$.

Example

Solve the initial value problem

$$(1+x)^2y'' + (1+x)y' - y = \log(1+2x+x^2), \;\; y(0) = 1 \; y'(0) = 3, \qquad (8)$$

on the interval $(-1, \infty)$.

Solution:

Let $1 + x = e^t \; or \; t = \log(1+x)$ then

$$(1+x)\frac{dy}{dx} = \frac{dy}{dt}; \;\; (1+x)^2\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Also, $log(1+2x+x^2) = \log(1+x)^2 = 2\log(1+x)$

Example

Then the given equation (8) becomes

$$\frac{d^2y}{dt^2} - y = 2t \tag{9}$$

The characteristic equation is

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Then the given equation (8) becomes

$$\frac{d^2y}{dt^2} - y = 2t \tag{9}$$

The characteristic equation is

$m^2 - 1 = 0 \implies m = 1, -1$

Thus, the complementary function $y_c(x)$ can be expressed as
$y_c(t) = (c_1 e^t + c_2 e^{-t})$

Example (Continued...)

$y_p(t) = At + B$

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Example (Continued...)

$y_p(t) = At + B$

$y_p'(t) = A$

$y_p''(t) = 0$

Substituting the value of $y_p(t), y_p'(t), y_p''(t)$ in (9)

$$-2A + At + B = t$$

Comparing the coefficients and solving we get, $A = -2,\ B = 0$.

Example (Continued...)

$\therefore y_p(t) = -2t,$

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$\therefore y_p(t) = -2t,$

the primitive is $y(t) = y_c(t) + y_p(t) = c_1 e^t + c_2 e^{-t} - 2t.$

$\therefore y(x) = c_1(1+x) + \frac{c_2}{(1+x)} - 2\log(1+x),$

$y'(x) = c_1 - \frac{c_2}{(1+x)^2} - \frac{2}{(1+x)}.$

Example (Continued...)

Using initial conditions $y(0) = 1$ *and* $y'(0) = 3$ we have

$$c_1 + c_2 = 1; \quad c_1 - c_2 - 2 = 3$$

solving which we get $c_1 = 3$ *and* $c_2 = -2$, and hence the solution of the initial value problem is

$$y(x) = 3(x+1) - \frac{2}{(1+x)} - \log(1+x)^2.$$

References I

- Martin Braun and Martin Golubitsky.
  Differential equations and their applications, volume 1.
  Springer, 1983.
- Richard Bronson.
  Schaum's outline of theory and problems of differential equations.
  McGraw-Hill, 1994.
- EA Coddington.
  An introduction to ordinary differential equations, phi learning pvt.
  Ltd., New Delhi, 2012.

References II

- BS Grewal.
  Higher engg. mathematics, 2006.
- George F Simmons.
  Differential equations with applications and historical notes.
  CRC Press, 2016.

Thank You