Quadrant II – Transcript and Related Materials

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Unit: 1

Module Name: Subgroups Example-Part III (Centralizer of an element of a Group)

Module No: 13

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Notes

Centralizer of a in G:

Let a be a fixed element of a group G. The centralizer of a in G, $C(a)$, is the set of all elements in G that commute with a.

In symbols,

$$C(a) = \{x \in G \mid ax = xa\}$$

Examples:

1. Centralizer of $a \in (U(n), \cdot_n)$, $C(a) = U(n)$.

   For every $a \in U(n)$, $a \cdot_n b = b \cdot_n a$, $\forall\ b \in U(n)$

   Hence, $C(a) = U(n)$.

2. Centralizer of $I_n \in GL(n, \mathbb{R})$, $C(I_n) = GL(n, \mathbb{R})$.

   $AI_n = I_nA \quad \forall\, A \in GL(n, \mathbb{R})$

   Hence, $C(I_n) = GL(n, \mathbb{R})$.

3. Centralizer of $i \in \mathbb{Q}_8$ is $\{1, -1\}$.

$\mathbb{Q}_8$ is a Quaternion Group defined as follows

$\mathbb{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

Here binary operation $\cdot$ is defined as

$$i^2 = j^2 = k^2 = -1, i \cdot j = k = -(j \cdot i), j \cdot k = i = -(k \cdot j), k \cdot i = j = -(i \cdot k),$$

$$1 \cdot x = 1 = x \cdot 1 \quad \forall\, x \in \mathbb{Q}_8, \quad -1 \cdot x = -x = x \cdot -1 \quad \forall\, x \in \mathbb{Q}_8$$

Therefore, $C(i) = \{1, -1\}$ and $C(-1) = \mathbb{Q}_8$

Properties:

- The centralizer of a in a group G is a subgroup of G.

Identity $e \in C(a)$ as $ea = ae, \quad \forall\, a \in G \Longrightarrow C(a) \neq \emptyset$.

Let $x, y \in C(a) \quad \Longrightarrow \quad xa = ax$ and $ya = ay$

$$\Longrightarrow y^{-1}yay^{-1} = y^{-1}ayy^{-1}$$

$$\Longrightarrow ay^{-1} = y^{-1}a$$

$$\Longrightarrow y^{-1} \in C(a).$$

Consider,

$$\begin{aligned}(xy^{-1})a &= x(y^{-1}a) \\ &= x(ay^{-1}) \\ &= (xa)y^{-1} \\ &= (ax)y^{-1} \\ &= a(xy^{-1})\end{aligned}$$

Hence, $xy^{-1} \in C(a)$.

Thus, by one step subgroup test, $C(a)$ is a subgroup of G.

- If G is an abelian group then for $a \in G$, $C(a) = G$.

$C(a) \subseteq G$.

Let $x \in G$ then $xy = yx \ \ \forall \, y \in G$ (since G is abelian)

In particular, $xa = ax$

$\implies \ x \in C(a)$.

Therefore, $G \subseteq C(a)$.

Hence, $C(a) = G$.

- For $a \in G, C(a)$ need not be an abelian group.

  Centralizer of $I_n \in GL(n, \mathbb{R}), \ \ C(I_n) = GL(n, \mathbb{R})$ (as shown in example 4)

  But $GL(n, \mathbb{R})$ is not abelian.

- Let G be a group then $Z(G) = \bigcap_{a \in G} C(a)$.

  Let $x \in \bigcap_{a \in G} C(a) \ \implies \ x \in C(a) \ \forall \, a \in G$

  $\implies \ xa = ax \ \ \forall \, a \in G$

  $\implies \ x \in Z(G)$

  $\implies \bigcap_{a \in G} C(a) \subseteq Z(G)$

  Now, let $x \in Z(G) \ \implies \ xa = ax \ \ \forall \, a \in G$

  $\implies x \in C(a), \ \forall \ a \in G$

  $\implies x \in \bigcap_{a \in G} C(a)$

  $\implies Z(G) \subseteq \bigcap_{a \in G} C(a)$

  Hence, $Z(G) = \bigcap_{a \in G} C(a)$.