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Module Name : Prime Ideals and Maximal Ideals

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Outline

- Introduction
- Prime Ideals
- Maximal ideals
- Examples
- References

Learning Outcomes

After studying this module you should be able to

1. define and identify prime ideals and maximal ideals.

Definition

A **proper** ideal P of a commutative ring R is called a **prime ideal** of R if whenever $ab \in P$ for $a, b \in R$, then either $a \in P$ *or* $b \in P$.

Example

$\{0\}$ *is a prime ideal of* Z *because* $ab \in \{0\} \implies a \in \{0\}$ *or* $b \in \{0\}$.

Example

Let R be an integral domain. Show that $I = \{(0, x) | x \in R\}$ is a prime ideal of $R \times R$.

Solution

Wkt I is an ideal of $R \times R$.
I is a proper ideal since $I \neq R \times R$.
To check I is proper ideal or not
Let $(a_1, b_1), (a_2, b_2) \in R \times R$ such that $(a_1, b_1)(a_2, b_2) \in I$.
Then $(a_1 a_2, b_1, b_2) = (0, x)$ for some $x \in R$.
$\therefore a_1, a_2 = 0, \ i.e., \ a_1 = 0 \ or \ a_2 = 0$, since R is a domain. Therefore, $(a_1, b_1) \in I$ or $(a_2, b_2) \in I$. Thus, I is a prime ideal.

Theorem

An ideal P of a commutative ring R with identity is a prime ideal of R if and only if the quotient ring R/P is an integral domain.

Proof.

Let us first assume that P is a prime ideal of R.
Since R has identity , so has R/P.
Now, let $a+P$ and $b+P$ bee in r/P such that $(a+P)(b+P)=P$, the zero element of R/P.
Then $ab+P=P$, *i.e.*, $ab \in P$. As P is a prime ideal of R either $a \in P$ or $b \in P$. So either $a+P=P$ *or* $b+P=P$.
Thus, R/P has no zero divisors.
Hence , R/P is an integral domain. $\square$

continued...

Conversely, assume that R/P is an integral domain. Let $a, b \in R$ such that $ab \in P$. Then $ab + P = P$ in R/P, *i.e.*, $(a+P)(b+P) = P$ *in* R/P is an integral domain, either $a + P = P$ or $b + P = P$, i.e. either $a \in P$ or $b \in P$.
This shows that P is a prime ideal of R. $\square$

Definition

A **proper** ideal M of a commutative ring R is called a **maximal ideal** if whenever I is an ideal of R such that $M \subseteq I \subseteq R$, then either $I = M$ or $I = R$.

Example

zero ideal in any field F is maximal ideal as only other ideal of F is F itself.

Theorem

Let R be a commutative ring with unity and let $M \subseteq R$ be an ideal. Show that R/M is a field if and only if M is a maximal ideal of R.

Proof.

First, suppose R/M is a field.
Define $\psi : R \to R/M$ by $\psi(r) = r + M$, which is easily seen to be a ring homomorphism.
Suppose we have $M \subseteq J \subseteq R$, where J is an ideal. Then $\psi(J)$ is an ideal of R/M , but since R/M is a field the only ideals are $\langle 0 + M \rangle$ and R/M .
If $\psi(J) = \langle 0 + M \rangle$ then we have $J \subseteq ker(\psi) = M \implies M = J$.
On the other hand, if $\psi(J) = R/M$ then we must have J = R .

$\square$

Continued...

To see this note that if for all $r \in R$ there is some $r_1 \in J$ such that $\psi(r_1) = r + M = \psi(r)$, then we must have $\psi(r - r_1) = (r - r_1) + M = 0 + M$.
But this means that $r - r_1 = m \in M \implies r = r_1 + m \in J$. Thus $R \subseteq J \implies J = R$. Therefore M is a maximal ideal.

Conversely, suppose M is a maximal ideal.
We already know that R/M is a commutative ring with unity, so all that remains is to show that every non-zero element has a multiplicative inverse.

$\square$

Continued...

Let $r + M \in R/M$ with $r \notin M$.
Now consider the ideal $\langle r + M \rangle inR/M$.
Since ψ is a homomorphism we know that $\psi^{-1}(\langle r + M \rangle)$ is an ideal of R .
Furthermore, this ideal properly contains M since $r \in \psi^{-1}(\langle r + M \rangle)$.
Since M is maximal, it must be the case that $\psi^{-1}(\langle r + M \rangle) = R$.
Thus $\psi(1) = 1 + M \in \langle r + M \rangle$, which means we can write $1 + M = (r + M)(s + M)$ for some $s \in R$. Hence r + M has a (multiplicative) inverse and so we conclude R/M is a field. $\square$

Remark

Maximal ideal is a prime ideal but converse is not true.

Example

- *The maximal ideals of* $\mathbf{Z}$ *are* $p\mathbf{Z}$*, here* p *is a prime.*
- *Let* $R = \mathbf{Z} \times \mathbf{Z}$*, Then* R *is a ring, where addition and multiplication defined by* $(a, b) + (c, d) = (a + c, b + d)$ *and* $(a, b).(c, d) = (ac, bd)$ *for all* $(a, b), (c, d) \in R$.
- *Let* $I = \mathbf{Z} \times \{0\}$*. Then* I *is not a maximal ideal of* R*. Since* $I \subset \mathbf{Z} \times 2\mathbf{Z} \subset R$.

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Thank You